A BENDING ANALYSIS OF SHALLOW THICK HYPERBOLICPARABOLOID SHELL BY RALEIGH RITZ METHOD

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SWAPAN LAHIRI

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'When we look at a thing we must examine its essence and treat the appearance merely as an usher at the threshold and once we cross the threshold, we must grasp the essence of the thing; this is the only reliable and scientific method of analysis.'

CERTIFICATE

Certified that the work A BENDING ANALYSIS OF SHALLOW THICK HYPERBOLICPARABOLOID SHELL BY RALEIGH RITZ METHOD has been carried out under my supervision and that has not been submitted elsewhere for a degree.

PDayaratnam

Dr. P. Dayaratnam
Associate Professor
Dept. of Civil Engineering
Indian Institute of Tech.,
Kanpur.

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To acknowledge all and sundry help will require volumes. And hence, practical limitations puts a restrain on me in doing that. Only a grateful appreciation is expressed to all such people who came out to extend friendly help, directly or indirectly, through criticism or by other means, providing thus a constant source of encouragement and enabling me to enliven up to the situation and thus complete the work. In spite of this summary expression a few names do pop up their head amongst these 'all'.

Not to mention of Dr. P. Dayaratnam would be tantamount to a grotesque violation of very rudimentary form of gratitude. But for his help this thesis might not have assumed this form.

A special mention of Shri R.K. Tewari, who helped me in a major part of the graphical work, and Shri J. K. Misra, whose typing has given the thesis an impressive look, need be made here to deride me, partly, of the debt I owe to them.

Swapan Lahiri

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

Hyperbolic paraboloid shells have good adoptability towards foundations. The ease in construction and the advantage of the shape for confinement of the soil enable the use of the hyper-shells for foundations in lose soils. Unlike the roof structures, the foundation shells are subjected to very high intensities of loads with more critical secondary conditions. Therefore, the foundation shells are relatively thick and call for bending analysis. The primary object of the thesis is to develop a qualitative study of shallow, and thick hypar shells and then suggest a range of moment coefficients for very simple boundary conditions. The total strain energy of the shell, including the shear deformation and normal stress is considered and the governing equations are developed by variational principle. Deflection coefficients are generated by Raleigh-Ritz's method for three different boundary conditions. The comparison of maximum deflection and bending moment of a simply supported shallow hypar shell is made with thin and thick plate theory results. As the rise of the shell decreases, the deflection and bending moments appear to tend towards the those of simple plate.

The deformation and bending behaviour of the shell with respect to different parameters such as span, height and thickness ratios, (the thickness ratio is defined as the ratio of the thickness to short span, similarly the other ratios), is studied. The

deflection and bending moment coefficients are found to be very sensitive and change rapidly for thickness ratio varying from 0.025 to 0.075. The variation of the coefficients damps and becomes asymptotic for the thickness ratios higher than 0.075. A series of deflection and bending moment coefficients are presented through a set of graphs. These coefficients can be used for design purpose of some idealised simple boundary conditions. Similar coefficients for different type of boundary conditions can easily be generated. A design example given at the end of the text illustrates the use of the graphs and suggest a method of design.

CHAPTER I

1.1 INTRODUCTION:

Theoretically, a shell may have any curved shape which is pleasing to the human feelings. However, the builders of the shells have been confined to a limited forms because of practical considerations. This last but very important factor speak heavily for the use of some 'anticlastic surfaces' which can be generated by straight lines and at the same time being clearly defined geometrically, |1|. Conoids, hyperboloids, and hyperbolicparaboloids belong to this classification of shells. The last variety of shells are the only warped surface whose equations are simple enough to permit stress calculations, |1|, with comparative ease.

Being a warped surface and easy to analyse, the hypar* shells have attracted many researchers to advance simplified analytic procedure |2, 3|. Since most of the shells used, till very recently, have been for roof structures (i.e. where loading is nominal), therefore they have been assumed as thin shells+.

This assumption was, in part responsible for the inhibited use of hypar shells as substructure, despite its clear superiority as load carrying member to 'plate' foundation (i.e. raft or mat

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* Hyperbolic paraboloids are commonly termed as hypar shells.

\+ A shell is said to be 'thin' if the thickness to minimum radius of curvature ratio can be neglected in comparison to unity |4|.

foundation). The nonavailability of a good theory for 'moderately thick' shells has restricted the use of hypar shells for foundation structures to a very limited extent |5|.

That the hypar shell foundation is far superior to ordinary mat foundation can be shown easily. The hypar shell foundation (a typical of which is shown in Fig. 1.1) has all the advantages as that of a mat namely,

(i) it provides an increased foundation area to develop minimum contact pressure and maximum safety factor against soil failure,

(ii) settlement will be minimum if the compressible soil strata is at a shallow depth. Moreover, this will be pronounced to a greater degree in case of a hyper shell due to the developed skin friction (Fig. 1.2). This skin friction being inclined has a vertical component and thus offers resistance to settlement. In case of a mat it is almost insignificant,

(iii) If there are small localised weak spots, the foundation can provide adequate safety against that.

(iv) Large hydrostatic lift can be resisted due to the heavy downward load. Furthermore, the continuity of the structure makes it possible to construct it soil flow tight.

One of the drawback with mat is that, the contact pressure distribution is nonuniform (Fig. 1.3)|6|. This cause large shear and bending stresses at the outer edges, particularly in case of a homogeneous, saturated clay. This defect can be obviated to a

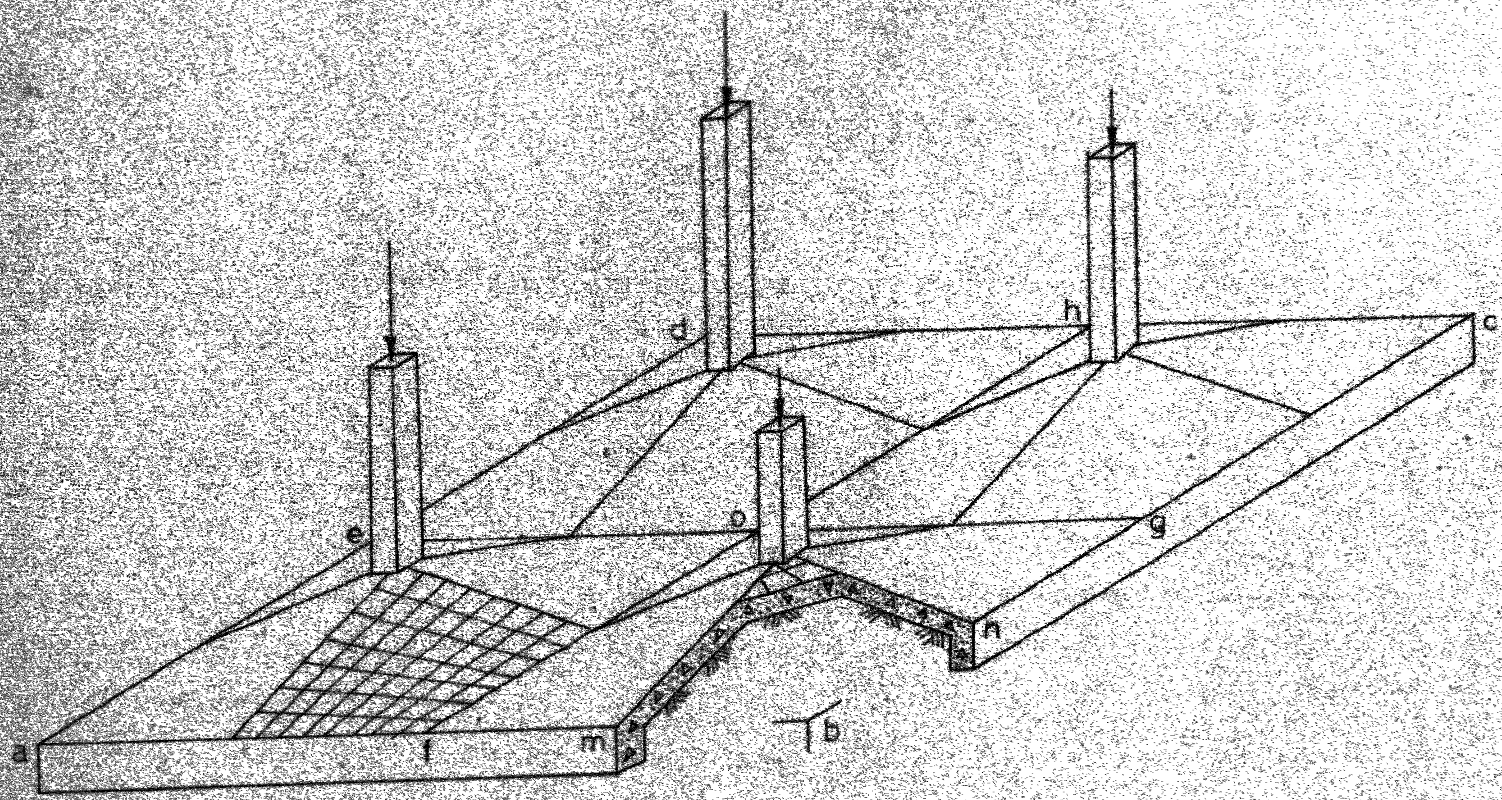


FIG. 1.1

A TYPICAL HYPER SHELL FOUNDATION

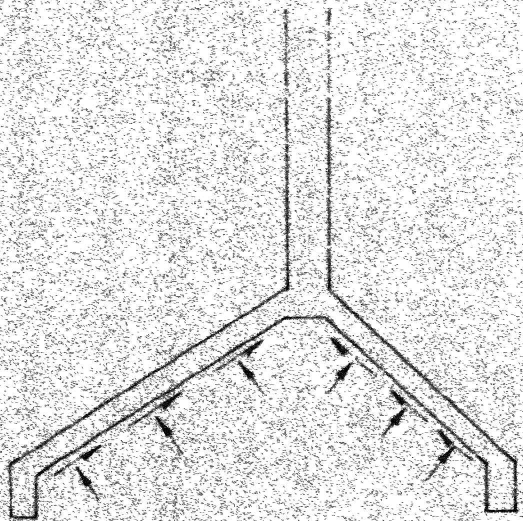


FIG 12

A SECTION OF THE SHELL FOUNDATION
SHOWING SKIN FRICTION

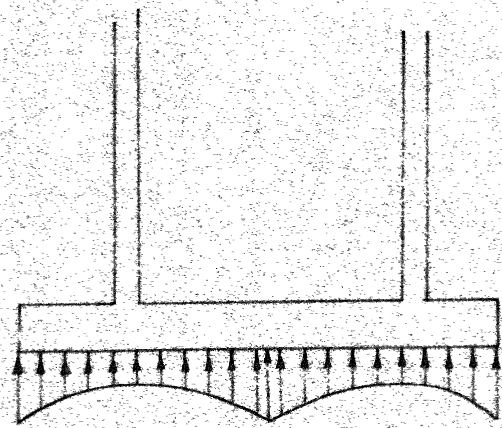


FIG 13

A TYPICAL MAT SHOWING CONTACT
PRESSURE DISTRIBUTION

large extent in a shell foundation, due to the mutual readjustment of skin friction forces. Again this more uniform pressure distribution prevents large concentration of bending and shear stresses at the outer edges and thus reducing the chances of 'flipping up' of the edges. A uniform pressure distribution along with skin friction makes its existence more welcome by lessening the chances of punching shear.

With all these in perspective a necessity was felt for providing a reasonably accurate procedure for design of such elements. And this work is only a beginning towards this goal without claiming to achieve a full proof design procedure.

1.2 OBJECT AND SCOPE:

This study is concerned with the bending analysis of shallow hyper shells with rectangular plane-form and bounded by the characteristics.

Since the motive force behind this work was to make available a set of data useful for actual design, many a simplifying assumptions have been made, which might give a feeling of nausea to a theoretician, but it was made only when it was felt that no great sacrifice of accuracy is involved. The price paid thus was well x worth considering the consequent ease of design.

Following this guideline an approximate solution by Raleigh-Ritz method was obtained. The results, for simply supported case, are presented in the form of graphs. The deflection function was chosen to satisfy the boundary conditions of the problems. Shells with more difficult boundary conditions can be solved by Galerkin

method wherein only the geometric boundary conditions are satisfied, but it was not attempted here.

Three boundary conditions were solved for, namely,

(i) when all edges are simply supported,

(ii) when two edges are clamped and two edges are simply supported,

(iii) when all edges are fixed.

The first case was investigated in detail, and the design charts given are for this case only. The two other cases were solved to check the feasibility of the method.

The simply supported case can be approximated in practice if,

(i) there are construction joints along eg and hf (Fig. 1.1),

(ii) it is assumed that the edges ab, bc, cd, da (Fig. 1.1) are supported by deep but narrow beams,

(iii) it is assumed that the lines joining the columns forms a sort of simple support (which is in fact the assumption for flat slab design).

1.3 SHELL ASSUMPTIONS:

The shells considered is shown in Fig. 1.4. They are doubly curved, shallow translational shells bounded by its characteristics and rectangular in planform. The middle surface of the shell, expressed in cartesian coordinates is given by:

$$Z(x, y) = kxy \tag{1.3.1}$$

where $k = c/ab$

and x and y are the characteristics of the hyperbola.

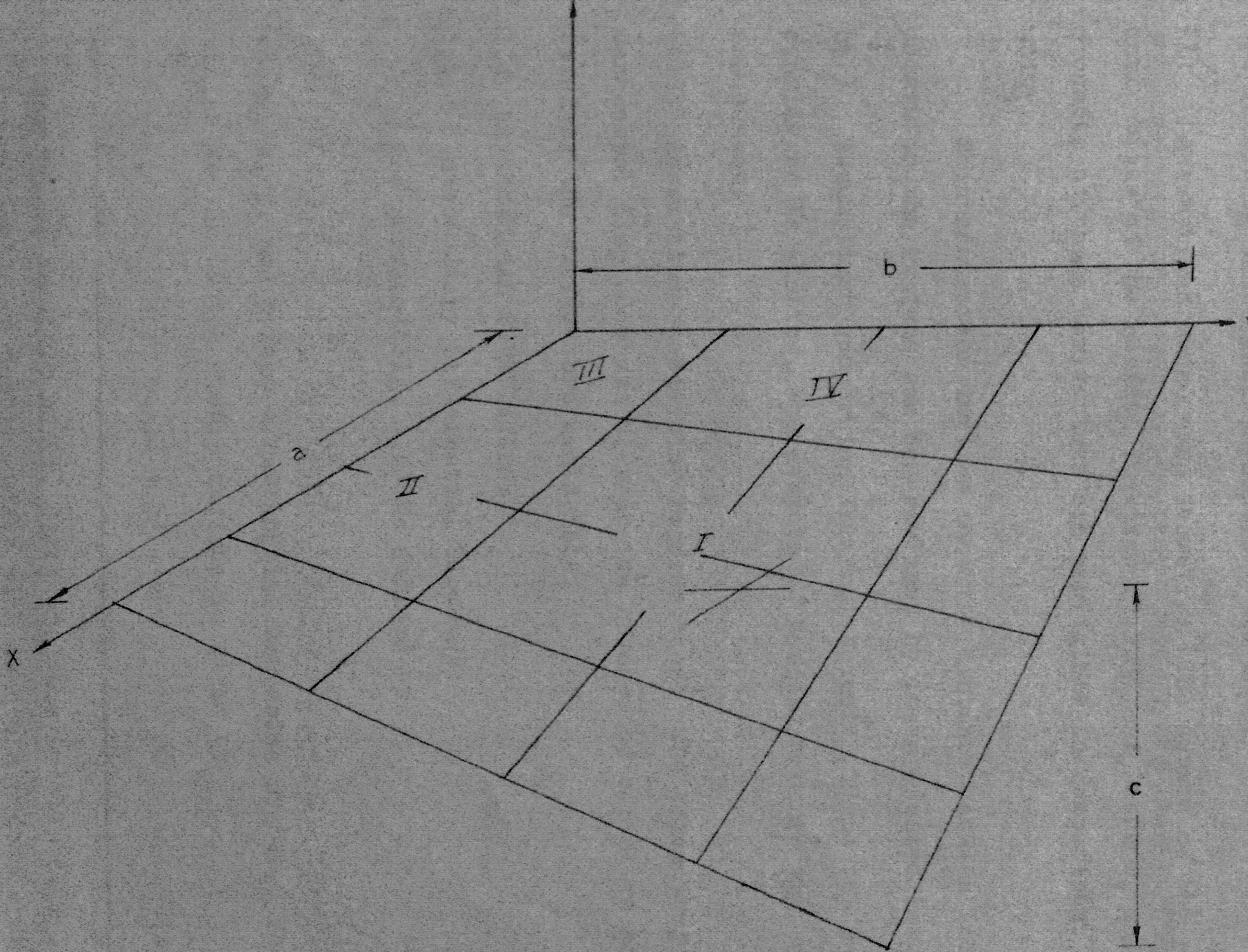


FIG. 1.4: HYPER SHELL SURFACE SHOWING THE DESIGN ZONES.

The shallowness is defined in the sense of Vlasov|7|. In this analysis besides the assumptions of the classical theory the following additional assumptions are made:

(i) Assumptions concerning Geometry: The square of the partial derivatives of the surface function (1.3.1), Z_x* and Z_y and the products of these derivatives are negligible in comparison with unity.

This assumption implies that the distance between two points on the surface can be replaced by its projected length.

(ii) Assumptions concerning Internal Forces: Since the shell considered is shallow, the effects of bending moments are well pronounced.

(iii) Assumptions concerning Deformations: The effects of the tangential displacements u, v on the changes of curvature and twist of the surface are negligible.

1.4 MATHEMATICAL FORMULATION:

1.4.1 Introductory Differential Geometry:

The surface (Fig. 1.4) is defined by

$$Z = Z(x, y) \tag{1.4.1}$$

where, x, y, and z form a rectangular cartesian system. The radius vector to the middle surface is given by

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \tag{1.4.2}$$

---

* For typing facility $\partial z/\partial x$, $\partial z/\partial y$ etc. are written as z_x, z_y respectively. A double suffix will denote a second order partial derivative.

or $$\bar{r} = x\bar{i} + y\bar{j} + z(x, y)\,\bar{k} \tag{1.4.3}$$

where $\bar{i}$, $\bar{j}$ and $\bar{k}$ are the unit vectors along x, y, and z respectively The equation to the tangent vectors s and t to the surface can be deduced as:

$$\begin{aligned} \bar{s} &= \frac{\partial \bar{r}}{\partial x} = \bar{i} + \frac{\partial z}{\partial x}\,\bar{k} \\ \bar{t} &= \frac{\partial \bar{r}}{\partial y} = \bar{j} + \frac{\partial z}{\partial y}\,\bar{k} \end{aligned} \tag{1.4.4}$$

The vectors sdx = ds and tdy = dt describe an elemental area on the shell surface. The element is not rectangular but its projection on the xy plane is. The vectors $\bar{s}$ and $\bar{t}$ are not unit vectors. The unit vectors $\bar{s}$, $\bar{t}$ and $\bar{\varsigma}$ (normal) direction are defined by $\bar{e}_s$, $\bar{e}_t$ and $\bar{e}_\varsigma$, where,

$$\begin{aligned} \bar{e}_s &= \frac{\bar{s}}{|\bar{s}|} = \frac{\bar{i} + (\partial z/\partial x)\,\bar{k}}{\sqrt{1+(\partial z/\partial x)^2}} \\ \bar{e}_t &= \frac{\bar{t}}{|\bar{t}|} = \frac{\bar{j} + (\partial z/\partial y)\,\bar{k}}{\sqrt{1+(\partial z/\partial y)^2}} \\ \bar{e}_\varsigma &= \bar{e}_s \times \bar{e}_t = \frac{\bar{k} - (\partial z/\partial x)\,\bar{i} - (\partial z/\partial y)\,\bar{j}}{\sqrt{1+(\partial z/\partial x)^2}\,\sqrt{1+(\partial z/\partial y)^2}} \end{aligned} \tag{1.4.5}$$

and the angle ω between $\bar{s}$ and $\bar{t}$ can be obtained as

$$\text{Cos}\,\omega = \frac{\bar{s}\cdot\bar{t}}{|\bar{s}|\;|\bar{t}|} = \frac{(\partial z/\partial x)\;(\partial z/\partial y)}{\sqrt{1+(\partial z/\partial x)^2}\,\sqrt{1+(\partial z/\partial y)^2}} \tag{1.4.6}$$

and,

$$\operatorname{Sin}\omega = \frac{\sqrt{1+(\partial z/\partial x)^2 + (\partial z/\partial y)^2}}{\sqrt{1+(\partial z/\partial x)^2}\,\sqrt{1+(\partial z/\partial y)^2}} \qquad (1.4.7)$$

The element of surface is obtained as

$$dA = \sqrt{1+(\partial z/\partial x)^2 + (\partial z/\partial y)^2}\; dx\, dy \qquad (1.4.8)$$

1.4.2 Pseudo-Stress Resultants:

A representative small element of the shell surface is formed by two radial planes whose horizontal lines are parallel to x-axis and the other radial planes of which the horizontal lines are parallel to y-axis, (Figs. 1.6 and 1.7).

The direct forces T_{xs} and T_{yt} are taken as positive when it creates tension. Similarly, the inplane shears T_{xt} and T_{ys} are positive when it creates tension in the diagonal direction of increasing values of x and y. $T_{x\varsigma}$ and $T_{y\varsigma}$ are the transverse shears.

The moments are positive when they produce tension in the top fibre. The positive twisting moments produce in plane shear stresses such that they result in diagonal tension along the increasing x and y direction.

All the forces and moments are expressed per unit length.

The force system in the shell element is reduced in terms of the force system in the plane element by resolving the vectors properly.

The relation between forces and stresses are (Fig. 1.5):

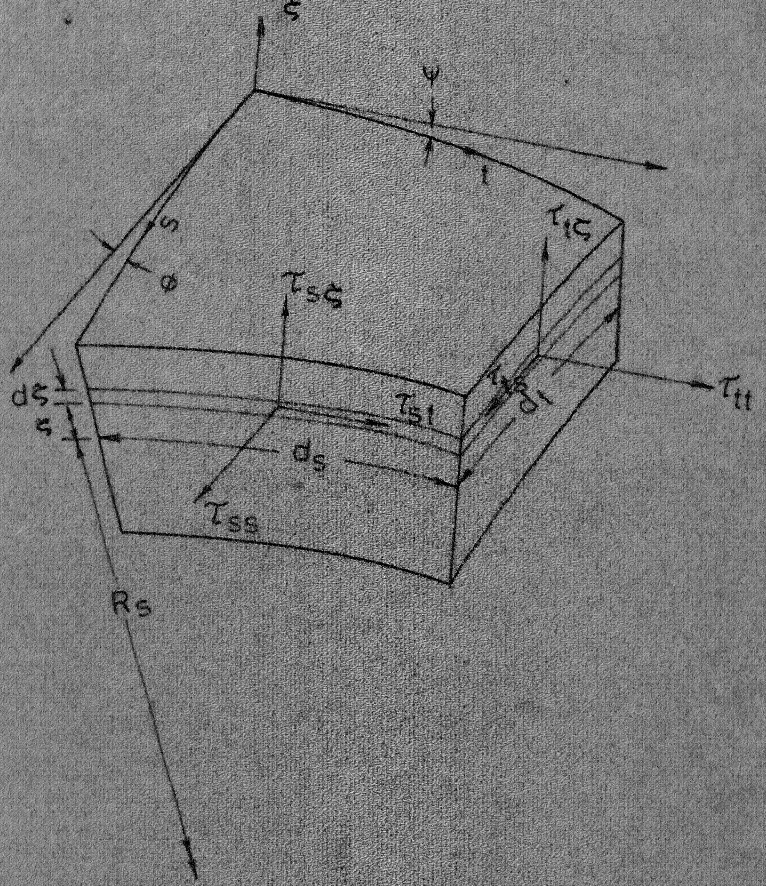


FIG. 1.5

FIG. 1.5: STRESSES AND ELEMENT NOTATIONS.

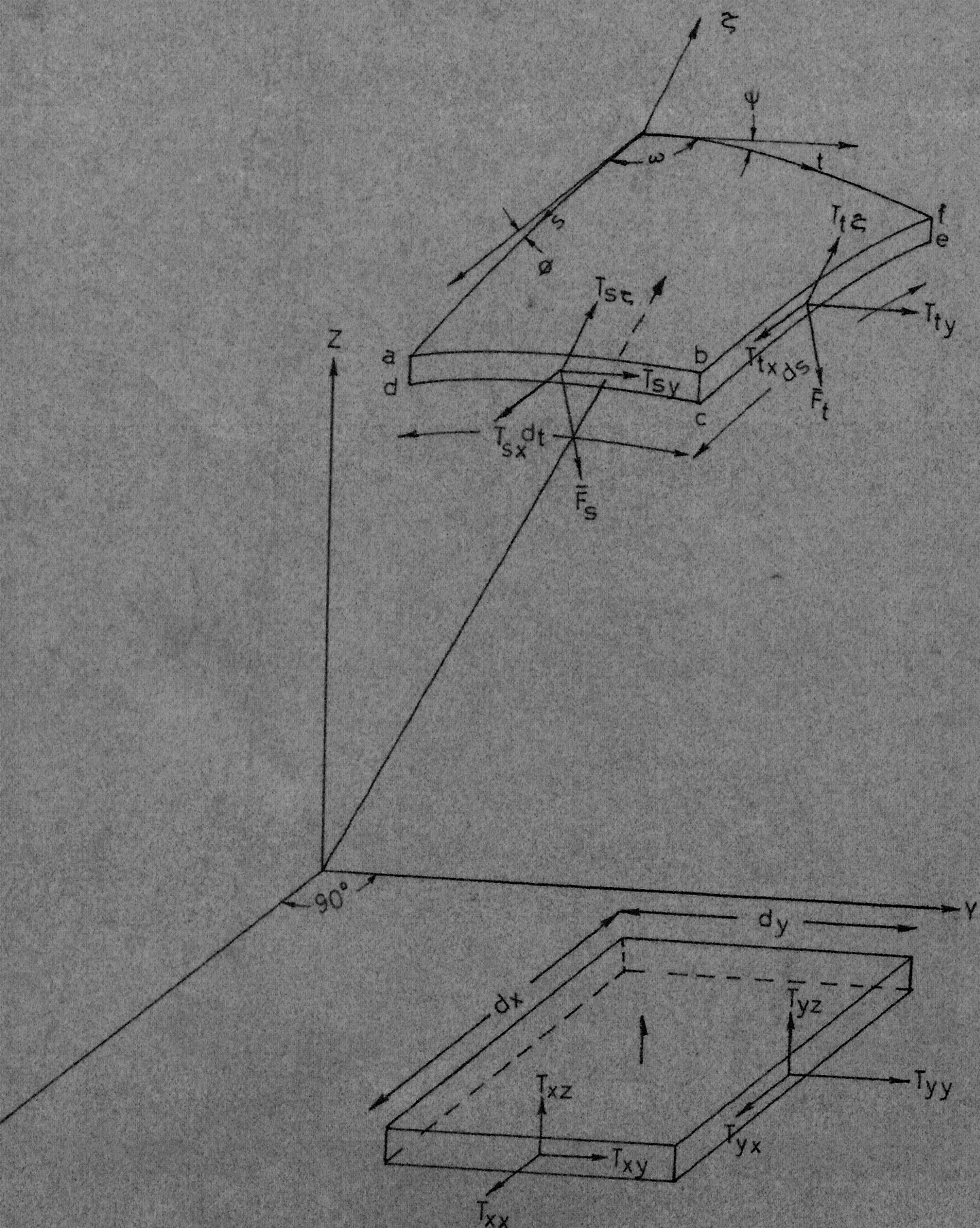


FIG. 1.6: EQUIVALENT STRESS RESULTANTS OF A SHELL ELEMENT

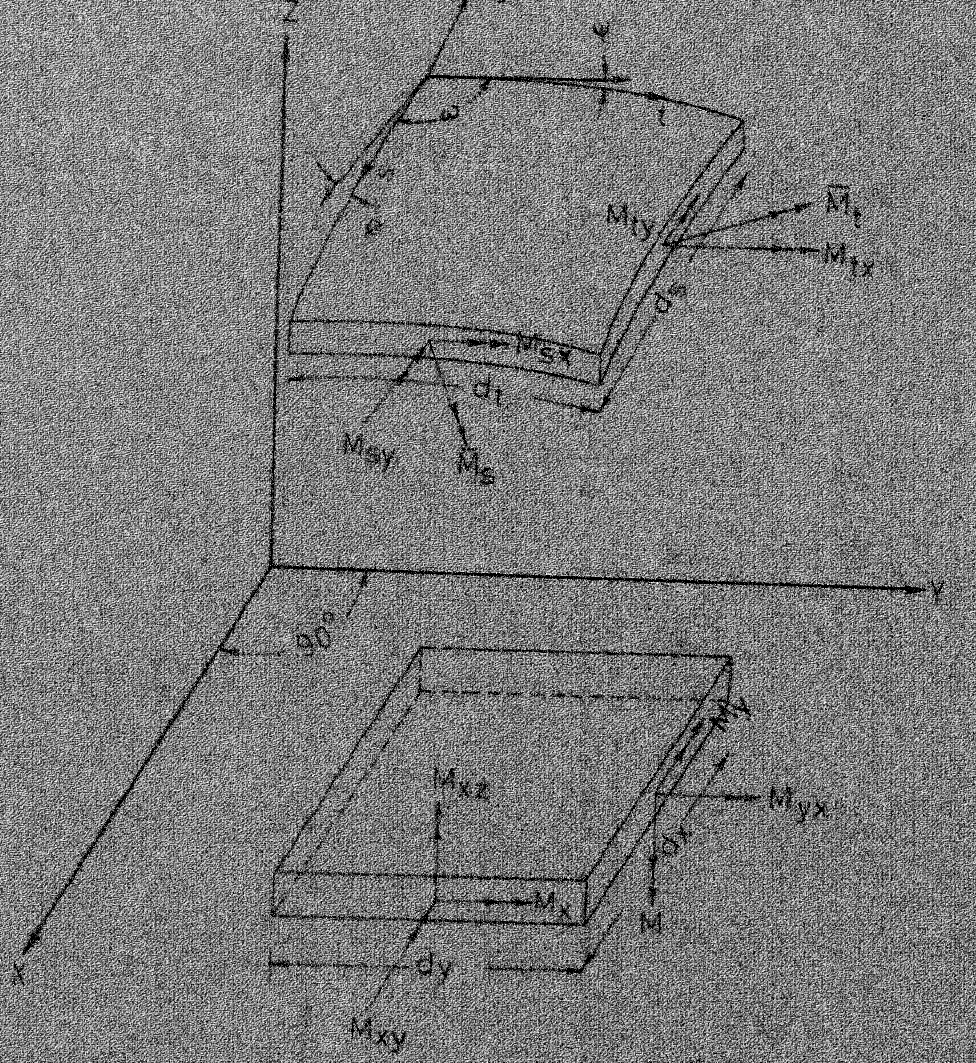


FIG. 1.7: EQUIVALENT STRESS RESULTANTS OF A SHELL ELEMENT.

FIG 17

$$T_{sx} = \int_{-h/2}^{+h/2} \tau_{ss} ((R_t + \zeta)/R_t) \, d\zeta$$

$$T_{sy} = \int_{-h/2}^{+h/2} \tau_{st} ((R_t + \zeta)/R_t) \, d\zeta \qquad (1.4.9)$$

$$T_{s\zeta} = \int_{-h/2}^{+h/2} \tau_{s\zeta} ((R_t + \zeta)/R_t) \, d\zeta$$

$$T_{ty} = \int_{-h/2}^{+h/2} \tau_{tt} ((R_s + \zeta)/R_s) \, d\zeta$$

$$T_{tx} = \int_{-h/2}^{+h/2} \tau_{ts} ((R_s + \zeta)/R_s) \, d\zeta \qquad (1.4.10)$$

$$T_{t\zeta} = \int_{-h/2}^{+h/2} \tau_{t\zeta} ((R_s + \zeta)/R_s) \, d\zeta$$

Similarly for moments we have

$$M_{sx} = \int_{-h/2}^{+h/2} \tau_{ss} \zeta ((R_t + \zeta)/R_t) \, d\zeta$$

$$M_{sy} = \int_{-h/2}^{+h/2} \tau_{st} \zeta ((R_t + \zeta)/R_t) \, d\zeta \qquad (1.4.11)$$

and,

$$M_{ty} = -\int_{-h/2}^{+h/2} \tau_{tt} \zeta ((R_s + \zeta)/R_s) \, d\zeta$$

$$M_{tx} = \int_{-h/2}^{+h/2} \tau_{ts} \zeta ((R_s + \zeta)/R_s) \, d\zeta \qquad (1.4.12)$$

In general T_{sy} and T_{tx} are not equal though τ_{st} and τ_{ts} are.

This is due to the inequality of the radii of curvature in s and t directions.

Let $\overline{F}_s$ and $\overline{F}_t$ be the resultant forces and $\overline{M}_s$ and $\overline{M}_t$ are the net moments (per unit length) acting on the two faces abcd and bcef of the shell element (Fig. 1.6 and 1.7). Then,

$$\overline{F}_s = T_{sx}\,\overline{e}_s + T_{sy}\,\overline{e}_t + T_{s\zeta}\,\overline{e}_\zeta$$

or,
$$\overline{F}_s = \frac{\overline{i} + (\partial z/\partial x)\,\overline{k}}{\sqrt{1+(\partial z/\partial x)^2}}\,T_{sx} + \frac{\overline{j} + (\partial z/\partial y)\,\overline{k}}{\sqrt{1+(\partial z/\partial y)^2}}\,T_{sy}$$

$$+ \frac{\overline{k} - (\partial z/\partial x)\,\overline{i}}{\sqrt{1+(\partial z/\partial x)^2}}\frac{- (\partial z/\partial y)\,\overline{j}}{\sqrt{1+(\partial z/\partial y)^2}}\,T_{s\zeta} \qquad (1.4.13)$$

Again, writing down the resultant force per unit length in terms of the forces in the plane element we have,

$$\overline{F}_s\,dt = (T_{xx}\,\overline{i} + T_{xy}\,\overline{j} + T_{xz}\,\overline{k})\,dy$$

or,
$$\overline{F}_s = (T_{xx}\,\overline{i} + T_{xy}\,\overline{j} + T_{xz}\,\overline{k})\,\frac{1}{\sqrt{1+(\partial z/\partial y)^2}} \qquad (1.4.14)$$

comparing (1.4.13) and (1.4.14) we have,

$$T_{xx} = \frac{\sqrt{1+(\partial z/\partial y)^2}}{\sqrt{1+(\partial z/\partial x)^2}}\,T_{sx} - \frac{(\partial z/\partial x)}{\sqrt{1+(\partial z/\partial x)^2}}\,T_{s\zeta}$$

$$T_{xy} = T_{sy} - \frac{(\partial z/\partial y)}{\sqrt{1+(\partial z/\partial y)^2}}\,T_{s\zeta} \qquad (1.4.15)$$

$$T_{xz} = \frac{(\partial z/\partial x)}{\sqrt{1+(\partial z/\partial x)^2}}\,T_{sx} + \frac{\partial z}{\partial y}\,T_{sy} + \frac{1}{\sqrt{1+(\partial z/\partial x)^2}}\,T_{s\zeta}$$

Writing $\sqrt{1+z_y^2} = K_y$, $\sqrt{1+(\partial z/\partial x)^2} = K_x$ and $\sqrt{1+(\partial z/\partial x)^2 + (\partial z/\partial y)^2} = Q$, the equations (1.4.15) can be rewritten as

$$T_{xx} = (K_y/K_x)\ T_{sx} - (1/K_x)\ (\partial z/\partial x)\ T_{s\varsigma}$$

$$T_{xy} = T_{sy} - (1/K_y)\ (\partial z/\partial y)\ T_{s\varsigma} \qquad (1.4.16)$$

$$T_{xz} = (K_y/K_x)\ (\partial z/\partial x)\ T_{sx} + (\partial z/\partial y)\ T_{sy} + (1/K_x)\ T_{s\varsigma}$$

Solution of T_{sx}, T_{sy} and $T_{s\varsigma}$ gives

$$T_{sx} = (K_y\ T_{xx} - \frac{(\partial z/\partial x)\ (\partial z/\partial y)}{K_y}\ T_{xy} + \frac{(\partial z/\partial x)}{K_y}\ T_{xz})\ (K_x/Q)$$

$$T_{sy} = (1/Q)\ (-(\partial z/\partial x)(\partial z/\partial y)\ T_{xx} + K_x^2\ T_{xy} + (\partial z/\partial y)\ T_{xz}) \qquad (1.4.17)$$

$$T_{s\varsigma} = -(K_x/Q)\ ((\partial z/\partial x)\ T_{xx} + (\partial z/\partial y)\ T_{xy} - T_{xz})$$

Similarly,

$$T_{yy} = (K_x/K_y) T_{ty} - ((\partial z/\partial y)/K_y)\ T_{t\varsigma}$$

$$T_{yx} = T_{tx} - ((\partial z/\partial x)/K_y)\ T_{t\varsigma} \qquad (1.4.18)$$

$$T_{yz} = (K_x/K_y)\ (\partial z/\partial y)\ T_{ty} + (\partial z/\partial x)\ T_{tx} + (1/K_y)\ T_{t\varsigma}$$

and,

$$T_{ty} = (K_x\, T_{yy} - ((\partial z/\partial x)(\partial z/\partial y)/K_x)\, T_{yx}$$

$$+ ((\partial z/\partial y)/K_x)\, T_{yz}\ )\ (K_y/Q)$$

$$T_{tx} = (1/Q)\ (-(\partial z/\partial x)(\partial z/\partial y)\ T_{yy} + K_y^2 + (\partial z/\partial x) T_{yz}) \quad (1.4.19)$$

$$T_{t\zeta} = -(K_y/Q)\ ((\partial z/\partial y)\ T_{yy} + (\partial z/\partial x)\ T_{yx} - T_{yz})$$

With the same argument as before the moment relations are obtained as (from Fig. 1.7),

$$\overline{M}_s = -M_{sy}\ \overline{e}_s + M_{sx}\ \overline{e}_t$$

or,

$$\overline{M}_s = -\frac{\overline{i} + (\partial z/\partial x)\ \overline{k}}{\sqrt{1+(\partial z/\partial x)^2}}\ M_{sy} + \frac{\overline{j}+(\partial z/\partial y)\ \overline{k}}{\sqrt{1+(\partial z/\partial y)^2}}\ M_{sx} \qquad (1.4.20)$$

In terms of the moments in the plane element, $\overline{M}_s$ can be expressed as:

$$\overline{M}_s = \frac{1}{\sqrt{1+(\partial z/\partial y)^2}}\ (-M_{xy}\ \overline{i} + M_{xx}\ \overline{j} + M_{xz}\ \overline{k}) \qquad (1.4.21)$$

Comparing (1.4.20) and (1.4.21),

$$M_{xx} = M_{sx}\ ,\quad M_{xy} = (K_y/K_x)\ M_{sy}$$

$$M_{xz} = -(K_y/K_x)(\partial z/\partial x)\ M_{sy} + (\partial z/\partial y)\ M_{sx} \qquad (1.4.22)$$

Similarly,

$$M_{yy} = M_{ty}\ ,\quad M_{yx} = (K_x/K_y)\ M_{tx}$$

$$M_{yz} = (\partial z/\partial x)\ M_{ty} - (K_x/K_y)\ (\partial z/\partial y)\ M_{tx} \qquad (1.4.23)$$

1.4.3. Stress Resultants for Shallow Shell:

Considerable simplification of equations (1.4.15) to (1.4.23) can be made by assuming the shell to be shallow |6|. This enables us to neglect terms $(\partial z/\partial x)^2$ and the like. Hence,

$$
\begin{aligned}
T_{sx} &= T_{xx} + (\partial z/\partial x)\, T_{xz} \\
T_{sy} &= (\partial z/\partial y)\, T_{xz} + T_{xy} \\
T_{s\zeta} &= -(\partial z/\partial x)\, T_{xx} - (\partial z/\partial y)\, T_{xy} + T_{xz}
\end{aligned}
\qquad (1.4.24)
$$

and,

$$
\begin{aligned}
T_{ty} &= T_{yy} + (\partial z/\partial y)\, T_{yz} \\
T_{tx} &= (\partial z/\partial x)\, T_{yz} + T_{yx} \\
T_{t\zeta} &= -(\partial z/\partial y)\, T_{yy} - (\partial z/\partial x)\, T_{xy} + T_{yz}
\end{aligned}
\qquad (1.4.25)
$$

and,

$$
\begin{aligned}
M_{xx} &= M_{sx} \\
M_{xy} &= M_{sy} \\
M_{xz} &= -(\partial z/\partial x)\, M_{sx} + (\partial z/\partial y)\, M_{sy}
\end{aligned}
\qquad (1.4.26)
$$

and,

$$
\begin{aligned}
M_{yy} &= M_{ty} \\
M_{yx} &= M_{tx} \\
M_{yz} &= -(\partial z/\partial y)\, M_{ty} + (\partial z/\partial x)\, M_{zx}
\end{aligned}
\qquad (1.4.27)
$$

The equations (1.4.24) to (1.4.27) give the forces and moments on the shell element in terms of those quantities on the plane element. This is done so, because analysis for a plane element is much more simpler than that for a curved element. To get the final results for the shell, the plane components

need these transformations which involve some algebraic manipulations only.

Before proceeding further, it will be advantageous if the equations to the surface is written out and the necessary modifications made. A hyperbolic paraboloid surface can be expressed as

$$z = (cxy/ab) \qquad (1.4.28)$$

where, x and y axes represent the characteristic of the hyperbola. Assuming the surface considerably simplieies the procedure since in this case we have $R_s = R_t = \infty$. Consequently, taking care of the cross-symmetry of the shear stresses we have

$$T_{sy} = T_{tx} = S$$

and (1.4.29)

$$M_{sy} = M_{tx} = M_t$$

1.4.4 Equilibrium Equations:

The equilibrium equations are written for the plane element.

$$(\partial T_{xx}/\partial x) + (\partial T_{xy}/\partial y) + q_x = 0$$

$$(\partial T_{yx}/\partial x) + (\partial T_{yy}/\partial y) + q_y = 0 \qquad (1.4.30)$$

$$(\partial T_{xz}/\partial x) + (\partial T_{yz}/\partial y) + q_z = 0$$

where,

$$\bar{q} = q_x \bar{i} + q_y \bar{j} + q_z \bar{k} \qquad (1.4.31)$$

and,

$$(\partial M_{xx}/\partial x) + (\partial M_t/\partial y) - T_{xz} = 0$$
$$(\partial M_t/\partial x) + (\partial M_{yy}/\partial x) - T_{yz} = 0 \qquad (1.4.32)$$

1.5 APPLICATION VARIATIONAL PRINCIPLE TO HYPAR SHELLS:

1.5.1 Total Potential:

To derive appropriate force-displacement relations the principle of minimum complementary energy with the notion of Lagrangian multiplier is used |10|. The functional

$$\pi = (1/2E) \int W \, . \, dv - \int \bar{F}.\bar{U} \, ds \qquad (1.5.1)$$

is variated subjected to the equilibrium relations to yield the appropriate force-displacement relations, where,

$$W = \sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 + 2(1+\nu)(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2) \qquad (1.5.2)$$

To obtain an analogus expression for the shell element in terms of the pseudo-stress resultants we consider a rectangular element (Fig. 1.8) on the middle surface of the shell. Let N_1, N_2, $N_{12} = T_{sy} = N_{21} = T_{tx} = S$, $N_{13} = T_{s\zeta}$ and $N_{23} = T_{t\zeta}$ and M_1, M_2, $M_{12} = M_{21} = M_{sy} = M_{tx} = M_t$ be the actual stress resultants acting over the faces of the element.

$$\therefore \int W \, dv = \int\int\int_{-h/2}^{+h/2} \Big(\big(\frac{N_1}{h} + \frac{12M_1}{h^3}\zeta\big)^2 + \big(\frac{N_2}{h} + \frac{12M_2}{h^3}\zeta\big)^2$$
$$-2\nu\big(\frac{N_1}{h} + \frac{12M_1}{h^3}\zeta\big)\big(\frac{N_2}{h} + \frac{12M_2}{h^3}\zeta\big)$$
$$+2(1+\nu)\Big\{\big(\frac{N_{12}}{h} + \frac{12M_{12}}{h^3}\zeta\big)^2 + \frac{9N_{13}^2}{4h^2}\big(1-\frac{4\zeta^2}{h^2}\big)^2$$
$$+\frac{9N_{23}^2}{4h^2}\big(1-\frac{4\zeta^2}{h^2}\big)^2\Big\}\Big) \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2} \, dx \, dy \, d\zeta \qquad (1.5.3)$$

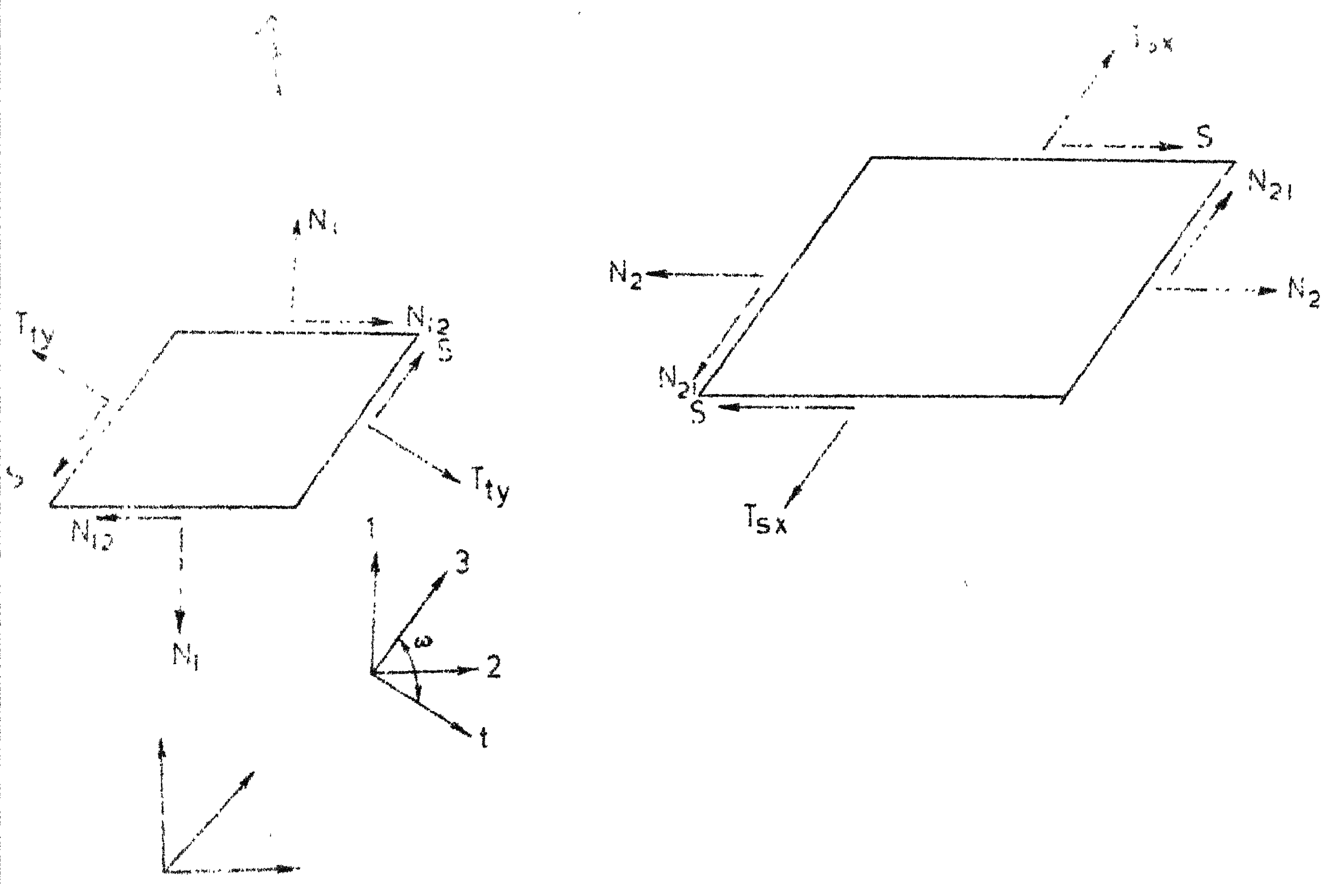


FIG 1.8 RECTANGULAR & PSENDO STRESS RESULTANTS ON THE ELEMENT OF SHELL SURFACE

Equation (1.5.3) can be rewritten, after carrying out the integration through the thickness, as

$$\int W\,dv = \frac{1}{h}\iint \left[(N_1+N_2)^2 + \frac{2(1+\nu)}{h}(N_{12}^2 - N_1N_2)\right] Q\,dxdy$$

$$+ \frac{12}{h^3}\iint \left[(M_1+M_2)^2 + 2(1+\nu)(M_{12}^2-M_1M_2)\right] Q\,dxdy$$

$$+ \frac{6}{5h}\iint (N_{13}^2 + N_{23}^2)\ Q\ dxdy \qquad (1.5.4)$$

Where the first double integral on the right hand side denotes membrane energy, the second denotes bending energy and the third part is the shear contribution to the total complementary energy of the shell element. It is this part which is of direct attention to us, about which more is discussed elsewhere in this work.

N_1, N_2 etc. are calculated in terms of the pseudo-stress resultants |7|. From Fig. (1.4), we have |7|

$$\begin{aligned}
N_1 \sin\omega &= T_{sx} + S\cos\omega \\
N_2 \sin\omega &= T_{ty} - S\cos\omega \\
N_{13} &= T_s \\
N_{23} &= T_t
\end{aligned} \qquad (1.5.5)$$

1.5.2: Total Potential of Shallow Shells:

Shallow shell approximation on the line of Vlasov enables to simplify the energy expressions considerably. A further simplification is necessary considering the enermity of the algebraic manipulations demanded for the solution. This constitutes in assuming that magnitude of shear stresses are small such that

when it is multiplied by quantities like $(\partial z/\partial x)$, $(\partial z/\partial y)$ etc, the resulting product is of higher order. This apparently 'innocent' looking assumption seems to be valid only in cases when the shell is 'shallow' as well as 'moderately thick'. Though no quantitative statement can be made at this stage about the natural questions 'how thick' and 'how shallow': but following Reissner |8| it can be said that as long as $\partial z/\partial x$ or $\partial z/\partial x$ does not exceed 1/3 and $h < 0.05$ times the smaller span these assumptions are good.

However, with this in view equations (1.3.24) and (1.3.25) can be rewritten as

$$T_{sx} = T_{xx} \quad , \quad T_{sy} = T_{xy} \quad \text{and} \quad T_{s\zeta} = -\frac{\partial z}{\partial x} T_{xx} - \frac{\partial z}{\partial y} T_{xy} + T_{xz}$$

and

$$T_{ty} = T_{yy} \quad , \quad T_{tx} = T_{yx} \quad \text{and} \quad T_{t\zeta} = -\frac{\partial z}{\partial x} T_{yx} - \frac{\partial z}{\partial y} T_{yy} + T_{yz}$$

and $$T_{sy} = T_{tx} = T_{yx} = T_{xy} = S \qquad (1.5.6)$$

with equations (1.5.6) and (1.5.5), keeping in view of the assumptions with reference to equation (1.4.6),

$$\begin{aligned}
(N_1 + N_2) &= T_{xx} + T_{yy} \\
N_1 \cdot N_2 &= T_{xx} \cdot T_{yy} \\
N_{12} &= S \qquad\qquad (1.5.7) \\
N_{13} &= -(\partial z/\partial x)\, T_{xx} - (\partial z/\partial y)\, S + T_{xz} \\
N_{23} &= -(\partial z/\partial x)\, S - (\partial z/\partial y)\, T_{yy} + T_{yz}
\end{aligned}$$

and,

$$M_1 + M_2 = M_{xx} + M_{yy}$$

$$M_{12} = M_t \qquad (1.5.8)$$

$$M_1 \cdot M_2 = M_{xx} \cdot M_{yy}$$

Now, the complementary energy expression (1.4.4) can be written as:

$$\int W\, dv = \frac{1}{h} \iint \left[(T_{xx}+T_{yy})^2+2(1+\nu)(S^2-T_{xx}\ T_{yy})\right] dx\ dy$$
$$+ \frac{12}{h^3} \iint \left[(M_{xx}+M_{yy})^2 + 2(1+\nu)(M_t^2-M_{xx}\ M_{yy})\right] dx\ dy$$
$$+ \frac{6}{5h} \iint \left[(T_{xz}^2 + T_{yz}^2)|-2T_{xz}(\partial z/\partial x)\ T_{xx} +(\partial z/\partial y)S)\right.$$
$$\left. - 2T_{yz}\ ((\partial z/\partial x)\ S + (\partial z/\partial y)T_{yy})\right] dx\ dy \qquad (1.5.9)$$

The equation (1.5.1) is variated with equation (4.5.9) in mind and taking care of the equilibrium equations (1.4.30) and (1.4.32) after being multiplied by λ's. Here one point should be noted that equation (1.4.32) is not an independent set of equations; it can be reduced with the help of (1.4.30c) and hence there are three independent equations out of the five as is apparent from (1.4.30) and (1.4.32) and so three values of λ, namely λ_1, λ_2 and λ_3 are required. The Lagrangian multipliers can be distinguished as u, v, w, respectively the three displacement functions |10| for the element.

Collecting like terms and then equated to zero, we have the force-displacement relations as:

$$(T_{xx} - T_{yy}) - \frac{12(1+\nu)}{5} \frac{\partial z}{\partial x} T_{xz} = Eh \frac{\partial u}{\partial x}$$

$$(T_{yy} - T_{xx}) - \frac{12(1+\nu)}{5} \frac{\partial z}{\partial y} T_{yz} = Eh \frac{\partial v}{\partial y} \qquad (1.5.10)$$

$$S - \frac{6}{5} \left(\frac{\partial z}{\partial y} T_{xz} + \frac{\partial z}{\partial x} T_{yz} \right) = \frac{Eh}{2(1+\nu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and,

$$(M_{xx} - M_{yy}) - \frac{h^2(1+\nu)}{10} \partial f_1/\partial x = - \frac{Eh^3}{12} \partial^2 w/\partial x^2$$

$$(M_{yy} - M_{xx}) - \frac{h^2}{10} (1+\nu) \partial f_2/\partial y = - \frac{Eh^3}{12} \partial^2 w/\partial y^2 \qquad (1.5.11)$$

$$M_t - \frac{h^2}{10} \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}\right) = - \frac{Eh^3}{6(1+\nu)} \partial^2 w/\partial x \, \partial y$$

where,

$$f_1(x,y) = T_{xz} - \frac{\partial z}{\partial x} T_{xx} - \frac{\partial z}{\partial y} S = T_{s\zeta} \qquad (1.5.12)$$

$$f_2 (x, y) = T_{yz} - \left(\frac{\partial z}{\partial x}\right) S - \left(\frac{\partial z}{\partial y}\right) T_{yy} = T_{t\zeta}$$

Equations (1.5.10) and (1.5.11) reduces to that obtained by Reissner for a plate |9| by setting $(\partial z/\partial x)$, $(\partial z/\partial y)$ etc. equal to zero, and boundary conditions are obtained as:

along x = constant

either,

$$T_{xx} = 0 \quad \text{or} \quad u - \bar{u} = 0$$

$$S = 0 \qquad v - \bar{v} = 0$$

$$T_{xz} = 0 \quad ; \quad w - \bar{w} = 0$$

$$M_{xx} = 0 \quad ; \quad -\left(\frac{\partial w}{\partial x}\right) + \frac{6(1+\nu)}{5Eh} f_1(x, y) = \theta_x \qquad (1.5.13)$$

$$M_t = 0 \quad ; \quad -\left(\frac{\partial w}{\partial y}\right) + \frac{6(1+\nu)}{5Eh} f_2(x, y) = \theta_y$$

where,

$\bar{u}$, $\bar{v}$ and $\bar{w}$ are the translation of the support, θ_x and θ_y are the slopes in x and y directions respectively along the edge x = constant. The boundary conditions along y = constant are similar.

CHAPTER II

SOLUTION BY RITZ METHOD

2.1 INTRODUCTION:

It can be shown |10| that the functions that minimise the complementary energy as given by equation (1.5.9) also determines the solutions of the appropriate Euler's equations, in this case the Euler's equations being the compatibility equations. Ritz method essentially consists in finding such a function in the form of an infinite series. The advantage of this form of the function lies in the fact that the series can be terminated after the desired degree of accuracy is achieved. The convergence of this sequence to the exact solution is usually a guaranteed, phenomenon, may be at a cost.

The method can be described, without going into mathematical polemics, by considering first the functional to be minimised as

$$I(u) = \iint_{\Sigma} f(x, y, u, u_x, u_y, \dots)\, dx\, dy \qquad (2.1.1)$$

where Σ spans the whole region under considerations.

Let the admissible solution $u^*(x, y)$ satisfies the boundary conditions $u = \phi(s)$, on the boundary, and makes I(u) a minimum. Now construct a sequence

$$u_n\ (x, y) = \phi_n\ (x, y, a_1, a_2, \dots, a_n) \qquad (2.1.2)$$

with parameters a_i. Then construct $I(u_n)$ with (2.1.2) and determine the parameters a_i such that $I(u_n)$ is a minimum, the $\sum_{n=0}^{\infty} u_n(x,y)$ forms a solution to the above problem. The sequence (2.1.2) is normally so chosen as to satisfy the prescribed boundary conditions.

The apparent docility of the method presents a too simple outlook, but in practice, the convergence criterion poses a stiff obstacle for numerical procedure. This is due to the fact that the tacit assumption about the completeness |11| of the sequence (2.1.2) is not always possible to achieve in practice. However, the loss of accuracy is not alarming compared to that already incurred in arriving at the energy expression, and thus lends credence for its use here.

2.2 POTENTIAL ENERGY IN TERMS OF DEFORMATION:

The equations (1.5.10) and (1.5.11) form a coupled set and hence their solution, if at all possible, involves excessive labor and mathematical sophistry. A practical way seems to be the minimisation of energy, expressed in terms of displacements. Here again, the two membrane displacements, u and v, can not easily be separated out from the displacement in the z-direction (w). This difficulty can be obviated if we assume that u and v are very small in comparison to w,* which is in fact the case for shallow shells. With this simplification in mind and the ordinary plate expressions for moments, we can write equations (1.5.9) in three parts as:

---

* This is discussed elsewhere in the work.

(i) Bending Energy:

$$U_b = \frac{D}{2} \iint \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left(\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \right) \right] dx\,dy$$

(ii) Shear Energy:

$$U_s = \iint \left[\frac{Eh^5(1+\nu)}{120(1-\nu^2)^2} \left\{ \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\, \partial y^2} \right)^2 + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right)^2 \right\} - \frac{Eh^3 K w}{5(1-\nu^2)} \left\{ \frac{\partial z}{\partial y} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\, \partial y^2} \right) + \frac{\partial z}{\partial x} \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \right\} \right] dx\,dy \tag{2.1.3}$$

(iii) Stretching Energy (due to w):

$$U_m = \frac{Eh\ k^2}{(1+\nu)} \iint w^2\ dx\ dy \tag{2.1.4}$$

where,

$$k = c/ab \tag{2.1.5}$$

Now the functional to be minimised becomes

$$I = \frac{1}{2} \iint \left[D \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left(\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \right) \right\} + K_1 \left\{ \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right)^2 + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right)^2 \right\} + K_2 w \left\{ \frac{\partial z}{\partial y} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) + \frac{\partial z}{\partial x} \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \right\} + K_3 w^2 - 2w \left(D \nabla^4 w - q_z \right) \right] dx\ dy \tag{2.1.6}$$

where,

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

and,

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$K_1 = \frac{Eh^3(1+\nu)}{60\,(1-\nu^2)^2}$$

$$K_2 = \frac{Eh^3 k}{10(1-\nu)} \qquad (2.1.7)$$

$$K_3 = \frac{Eh\,k^2}{2(1+\nu)}$$

CHAPTER III

3.1 SOLUTIONS FOR PARTICULAR CASES:

The displacement function (w) is assumed, satisfying the boundary conditions for the case under considerations. With this displacement function the expression (2.1.6) is written out explicitly. The first derivative of the integral, with respect to the undetermined constant in w, is then equated to zero to yield the value of the constant for which I is minimum.

3.1.1 All edges are simply supported:

The displacement function assumed in this case is the usual double sine series:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \operatorname{Sin} \frac{n\pi x}{a} \operatorname{Sin} \frac{m\pi y}{b} \qquad (3.1.1)$$

Putting this value of w in equation (2.1.6) and equating $\frac{\partial I}{\partial A_{mn}}$ to zero, the value of A_{mn} is obtained as

$$A_{mn} = \frac{32\,(qo/D)b^4}{(2m+1)(2n+1)\pi^2} \Bigg/ \Bigg[\pi^4\Big\{(2m+1)^2+\alpha^2(2n+1)^2\Big\}^2 \left(\frac{\gamma^2\pi^2}{5(1-\nu)}\left((2m+1)^2+\alpha^2(2n+1)^2\right)-1\right) + 6\alpha^2\beta^2\left(\frac{(1-\nu)}{\gamma^2} + \frac{\pi^2}{5}\left((2m+1)^2 + \alpha^2(2n+1)^2\right)\right)\Bigg] \qquad (3.1.2)$$

where,

$$\alpha = a/b, \quad \beta = c/b, \quad \gamma = h/b \qquad (3.1.3)$$

and,

a = Larger Span

b = Smaller Span

c = Rise of the shell

h = Thickness of the shell

q_o = Uniformly distributed load

3.1.2 Two Opposite Edges are Simply Supported and Other Two Edges are Fixed:

The assumed displacement function is

$$w = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left(1 - \cos \frac{m\pi y}{b}\right) \sin \frac{n\pi x}{a} \tag{3.1.4}$$

and A_{mn} is obtained as

$$\begin{aligned} A_{mn} = \frac{2q_o}{D} b^4 \Big/ \Big[& 4\alpha^2 m^2 n^2 \pi^2 + \\ & - \frac{\gamma^2 \pi^6}{5(1-\nu)} \left\{ \left(m^6 + 3\alpha^4 m^2 n^4 + 3\alpha^2 m^4 n^2 + \alpha^6 n^6 \right) \right\} \\ & + \frac{6(1+\nu)}{5} \alpha^2 \beta^2 \pi^2 \left((3m^2 + \alpha^2 n^2)\left(1 + 2(-1)^n\right) \right) \\ & - 18(1-\nu)\alpha\beta \Big] \end{aligned} \tag{3.1.5}$$

3.1.3: All Edges are Fixed:

The displacement function is assumed to be

$$w = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left(1 - \cos \frac{n\pi x}{a}\right) \left(1 - \cos \frac{m\pi y}{b}\right) \tag{3.1.6}$$

Proceeding as before, A_{mn} is obtained as:

$$A_{mn} = \frac{4q_o}{D} b^4 \Big/ \Big[(3\pi^4 (2m+1)^4 \left(1 + \frac{4}{(2m+1)\pi}\right) + 3\alpha^4\pi^4(2n+1)^4 \left(1+\frac{4}{(2n+1)\pi}\right) + \frac{3\gamma^2\pi^6}{5(1-\nu)} ((2m+1)^6 + \alpha^6(2n+1)^6 + \alpha^2 (2n+1)^2(2m+1)^2 ((2m+1)^2+\alpha^2(2n+1)^2)) -6\alpha\beta^2\pi^2 (2\alpha (3(2m+1)^2 + \alpha^2(2n+1)^2) + 2((2m+1)^2 + 3\alpha^2(2n+1)^2)) + 54(1-\nu)\alpha^2\beta^2/\gamma^2 \Big] \qquad (3.1.7)$$

3.2 ACCURACY OF THE SOLUTIONS:

The accuracy of the equations (1.5.10) to (1.5.12) were discussed before.

The numerical values obtained from (3.1.1) are resubstituted in (1.5.11) and the residue (the difference between the two sides of the equality sign) was calculated as a percentage of the maximum moment and was found to be insignificantly small.

For $\beta = 0$ (i.e. the shell reduced to a plate) the moments and deflections were checked with Timoshenko |12| and Goldberg|15| as is shown in the accompanied graphs of moment coefficients in the central zone (zone I) and are found to be in well agreement.

For a shell with $\alpha = 1.5$, $\beta = 0.25$ and $\gamma = 0.025$ moments and deflections were calculated along the centre of the span in the shorter direction. These were then compared with that obtained by Bouma's |13| approach and are shown in Fig. 3.1.

For a shell with $\alpha = 1.0$, $\beta = 0.3$ and $\gamma = 0.02$, $w = 3000$ Kg, Dayaratnam et al |44| obtains

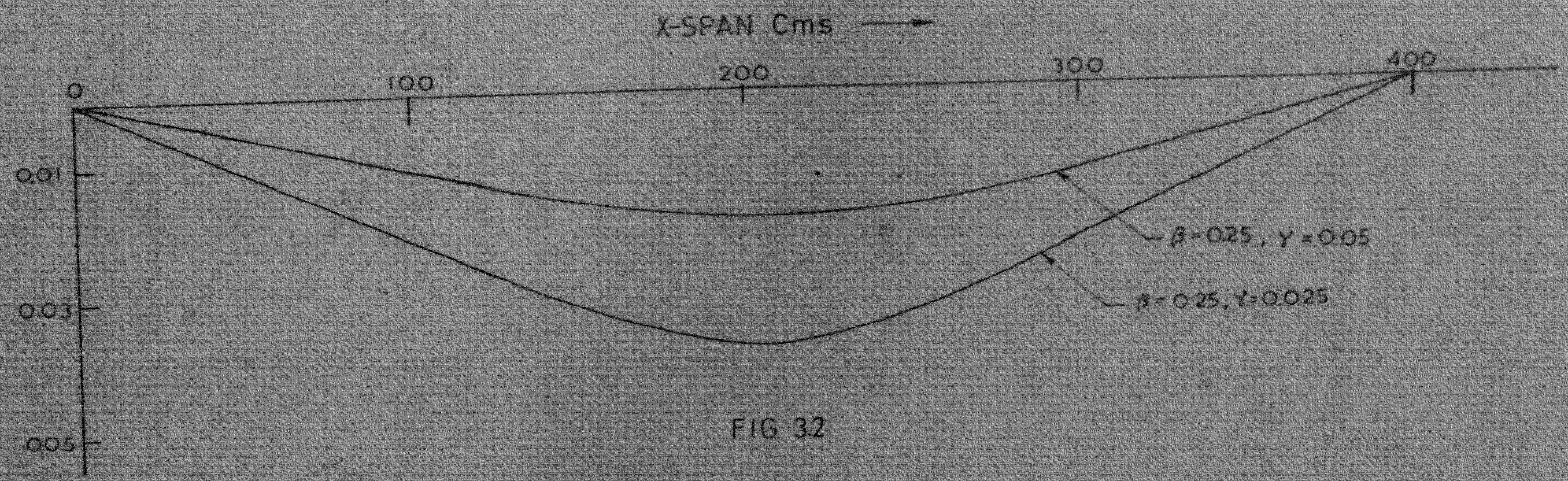


FIG 3.2

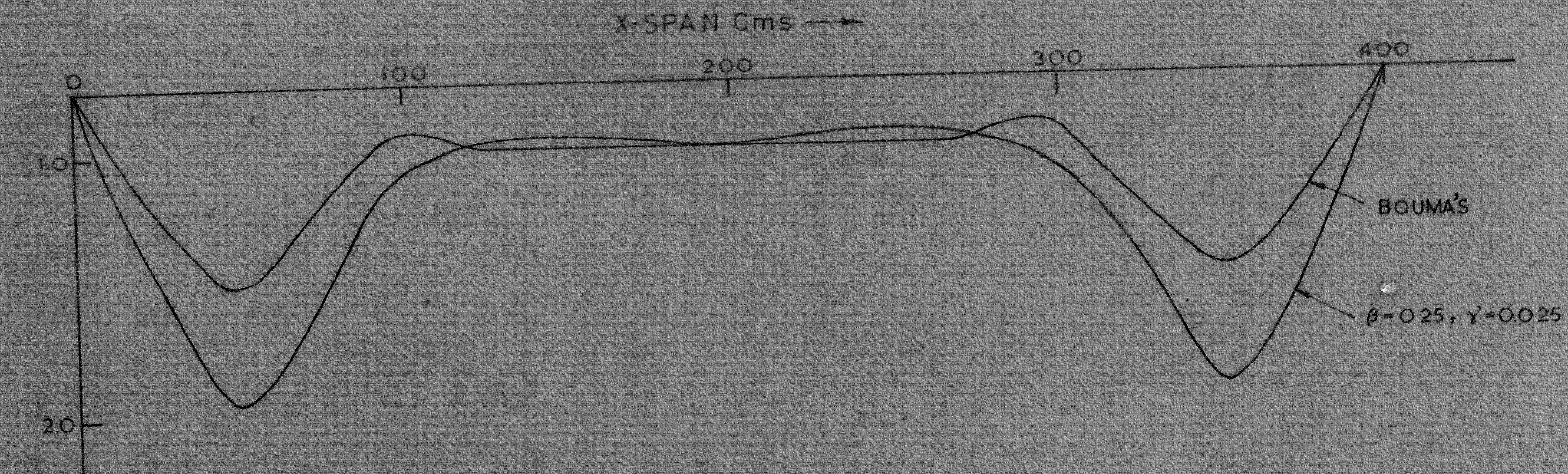


FIG. 3.1

FIG. 3.2 & 3.3 DEFLECTION AND MOMENT DISTRIBUTION ALONG THE CENTRE OF THE LONGER SPAN FOR A SIMPLY SUPPORTED HYPER SHELL, BOUNDED BY ITS CHARACTERISTICS.

w_{max} = 2 mm.

$M_{xx_{max}}$ = 8 Kg. cm.

According to the present investigation

w_{max} = 3.27 mm

$M_{xx_{max}}$ = 4.15 Kg. cm.

1.3.3 Discussion of the Design Graphs:

Three aspect ratios have been defined as

$\alpha = a/b$, the Span ratio

$\beta = c/b$, the rise ratio

$\gamma = h/b$, the thickness ratio.

Moments and deflection coefficients have been plotted for various values of α, β, γ . These deflection coefficients refer to the deflection of the middle point of the shell where the maximum deflection occurs. These coefficients are non-dimensionalised parameters and the exact relations for moments and deflections are,

$$w = \delta \, (q/E) \, b \qquad (3.3.1)$$

$$M_{Ix} = \mu_{Ix} \, qa^2 \qquad (3.3.2)$$

$$M_{Iy} = \mu_{Iy} \, qb^2 \qquad (3.3.3)$$

The subscripts in equations (3.3.2) and (3.3.3) refers to the zone number and direction of the vector respectively. To facilitate the design and placing of reinforcements, the shell is

divided into nine zones (Fig. 1.4) of which four are typical. For each value of span ratio, the curves are the variation in moment coefficients for different values of thickness ratios (γ) and for a particular zone and direction, the corresponding rise ratios (β) are also specified therein.

The apparent strangeness in presenting only the bending moments without a mention of the membrane forces can be explained on two grounds:

1. The effect of membrane forces in a thick shell is neglegible in the sense that the minimum allowable reinforcements (as is prescribed by IS code) far exceeds that obtained from an actual membrane stress calculation.

2. From an investigation |17| it is inferred that for $\beta \gamma/\alpha$ ratio lying within 0.025 to 0.001 and β varying between 0.1 and 0.25 the moments predominates to a high degree. The cases investigated in this work fall in this category and thus substantiates the prescription of a design method based on bending moments.

3.4 DESIGN PROCEDURE:

The design procedure can best be illustrated by an example. The superiority of the shell foundation can be seen from the comparison of the two designs presented below, the first one is for that of a conventional footing and the second one is the shell foundation which is advocated in this work.

Example: Suppose a footing has to be designed for a 25 cm x 25 cm. square column carrying a load of 25 tonnes in a

soil of bearing capacity of 1.0 T / M^2 (or 0.1 Kg/cm^2).

Conventional design: (Fig. 3.35)

Superimposed load = 25 tonnes

Footing load (10 percent) = 2.5 Tonnes

Total vertical load 27.5 Tonnes

Area required = 27.5/1.0 = 27.5 m^2

Provide: $5.5^m \times 5.5^m = 30.25\ m^2$

Actual soil pressure = 27.5/30.25 = 0.91 T/M^2 < 1 T/M^2 (Safe).

$$\text{Max.}^m \text{ bending moment} = (1/24) \times 0.91\,(2 \times 5.5 + 0.25)\,(5.5 - 0.25)^2$$

$$= 11.75 \times 10^5 \text{ Kg cm}$$

$$\text{Depth required} = \sqrt{(11.75 \times 10^5)/(25 \times 9.7)} = 48.5 \text{ cm.}$$

Depth provided at the base of the column = 50 cm. (See Fig.3.36)

$$\text{Intensity of Punching shear} = (1/4) \times \frac{0.91\,(5.5^2 - 0.25^2)}{0.15 \times 0.485}$$

$$= 5.66 \text{ Kg/cm}^2 < 10.5 \text{ Kg/cm}^2$$

(Safe)

Depth provided at the edge = 15 cm. (See Fig. 3.36)

Effective depth at the critical section = 35.7 cm.

$$\text{Shear force} = \frac{910 \times 6.75 \times 2.125}{2} = 6520 \text{ Kg}$$

$$\text{Shear stress} = \frac{6520 \times 7}{125 \times 6 \times 50} = 1.16 \text{ Kg/cm}^2 \quad \text{(Safe)}$$

$$\text{Area of steel} = \frac{11.75 \times 10^{5} \times 7}{1265 \times 6 \times 48.5} = 52.5 \text{ Sq. cm.}$$

Use 16 mm ϕ at the rate of 20 cm. c/c. (See Fig.3.36)

Shell Design:

The area of the shell in plan is assumed to be the same. since this is governed by the allowable bearing pressure.

For this case,

Span Ratio,

$$\alpha = 5.5/5.5 = 1.0$$

a uniform shell thickness of half that of the 'plate' is assumed

So thickness provided = 25 cm.

$\therefore$ thickness ratio, $\gamma = 25/550 = 0.05$

Figs. 3.8 - 3.10 and Fig. 3.4 gives the moment and deflection coefficients for the span ratio of 1.0.

Though a greater rise will reduce the bending moments considerably but to show the effectiveness of the method only 10 percent rise (i.e. $\beta = 0.1$) is assumed.

From the above graphs, and for these values of β and γ, the following values are obtained

$$\delta = 300, \quad \mu_1 = 5.23, \quad \mu_{II} = 5.52, \quad \mu_{III} = 4.94$$

Corresponding moments:

$$M_I = 15.90 \times 10^3 \text{ Kg cm}$$
$$M_{II} = 16.70 \times 10^3 \text{ Kg cm}$$
$$M_{III} = 15.00 \times 10^3 \text{ Kg cm}$$

Maximum depth required $= \sqrt{\dfrac{16.70 \times 10^3}{25 \times 9.7}} = 8$ cm < 20 cm (Safe)

But so small a depth is not feasible considering the punching shear.* With the assumed depth of 20 cm, the intensity of punching shear is obtained as 11.66 Kg/cm^2 (10.5 Kg/cm^2 and hence safe).

Area of steel required will be very small considering the ratio of maximum bending moments in the his cases. Hence, the minimum permissible steel is to be provided (i.e. 10 mm Ø at the rate of 25 cm. c/c).

Above two cases clearly show the superiority of shell foundation over ordinary spread foundation.

3.5 DISCUSSION OF RESULTS:

The deflection coefficients (δ) decreases very rapidly as the thickness ratio (γ) increases from 2.5 percent to 5 percent (Fig. 3.4 - 3.7). This decreement rate then becomes asymptotic for further increase of thickness ratio. For 10 percent rise of the shell (i.e. $\beta = 0.1$) the deflection coefficient reduces to 1/5 to 1/8 for thickness ratio varying between the above two limits; the later values are for higher span ratios (i.e. $\alpha > 1.0$).

With only a 10 percent rise the shell deflection coefficient reduces considerably from the plate values (See Fig. 3.6 and 3.7). Increasing the shell rise also reduces the deflection

---

* Punching shear in a shell will be smaller due to the vertical component of the inclined membrane forces, but this effect was not considered here for simplicity.

coefficient (δ) considerably for the thin shell range. For 25 percent shell rise the δ values are almost half of the δ values for 10 percent rise. But the decreements are not so much observed for higher thicknesses; in fact, for 7.5 percent or more thickness ratio the shell rise has no perceptible effect on deflection coefficients. This later observation can be explained from the point of view that with increase percentage thickness the curvature effect of the shell decreases and hence its 'shell superiority over the plate is decreased reducing it almost to an inclined plate

That with increased thickness the shell starts to behave almost as a plate, particularly for small rise to span ratio, is also corroborated by the curves for moment coefficients. For example, all the curves show little perceptible difference between plate moment values and that for shells near the 10 percent thickness range, though for thin shell values a wide difference is observed, as is expected.

A look at Fig. 3.8 to 3.34 shows that a 10 percent rise reduces the moment coefficients by at least 30 percent from the plate values. This reduction normally lies between 30 percent to 50 percent. With increase in shell rise the moment coefficients reduces considerably; this effect is particularly pronounced near the central zone (zone I, Fig. 1.4).

A comparison of Fig. 3.11 and Fig. 3.15 shows that increasing the span ratios (γ) decreases the moment in the longer direction. This effect is particularly pronounced for $\alpha > 1.5$ and near

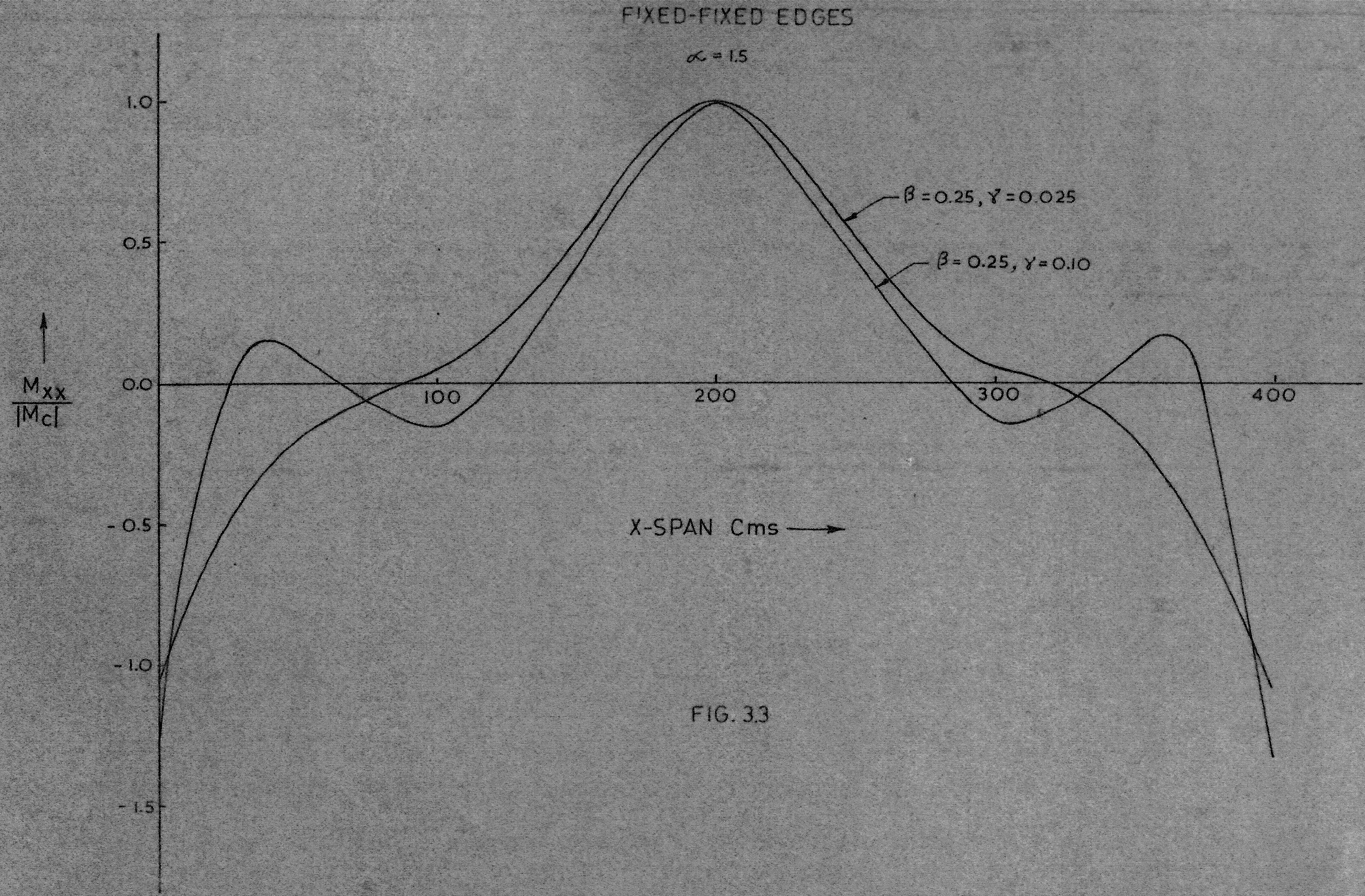


FIG. 3.3

FIG.3.3: BENDING MOMENT DISTRIBUTION ALONG THE CENTRE OF THE LONGER SPAN FOR A SHELL WITH FIXED FIXED EDGE CONDITIONS.

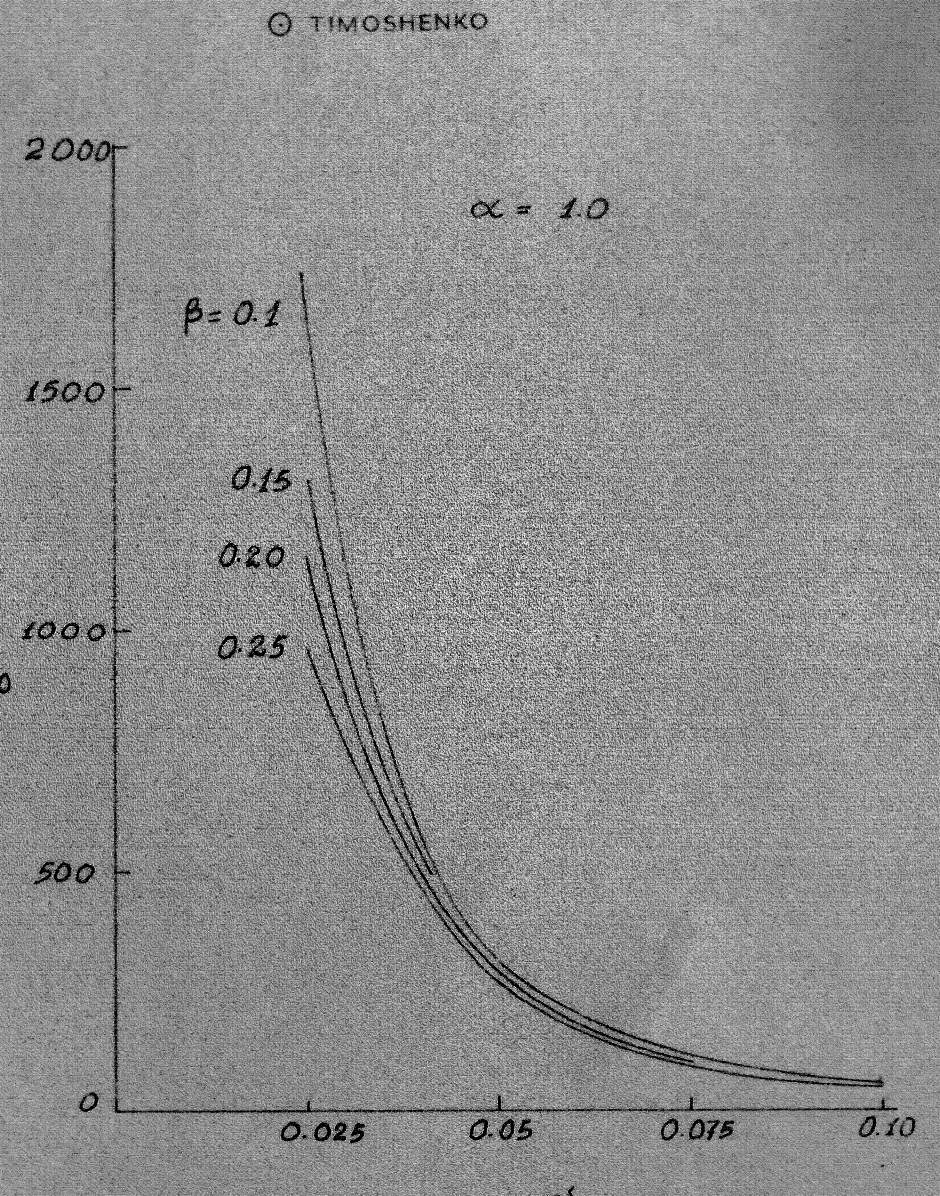


FIG. 3.4: DEFLECTION COEFFICIENTS FOR SIMPLY SUPPORTED SHELL.

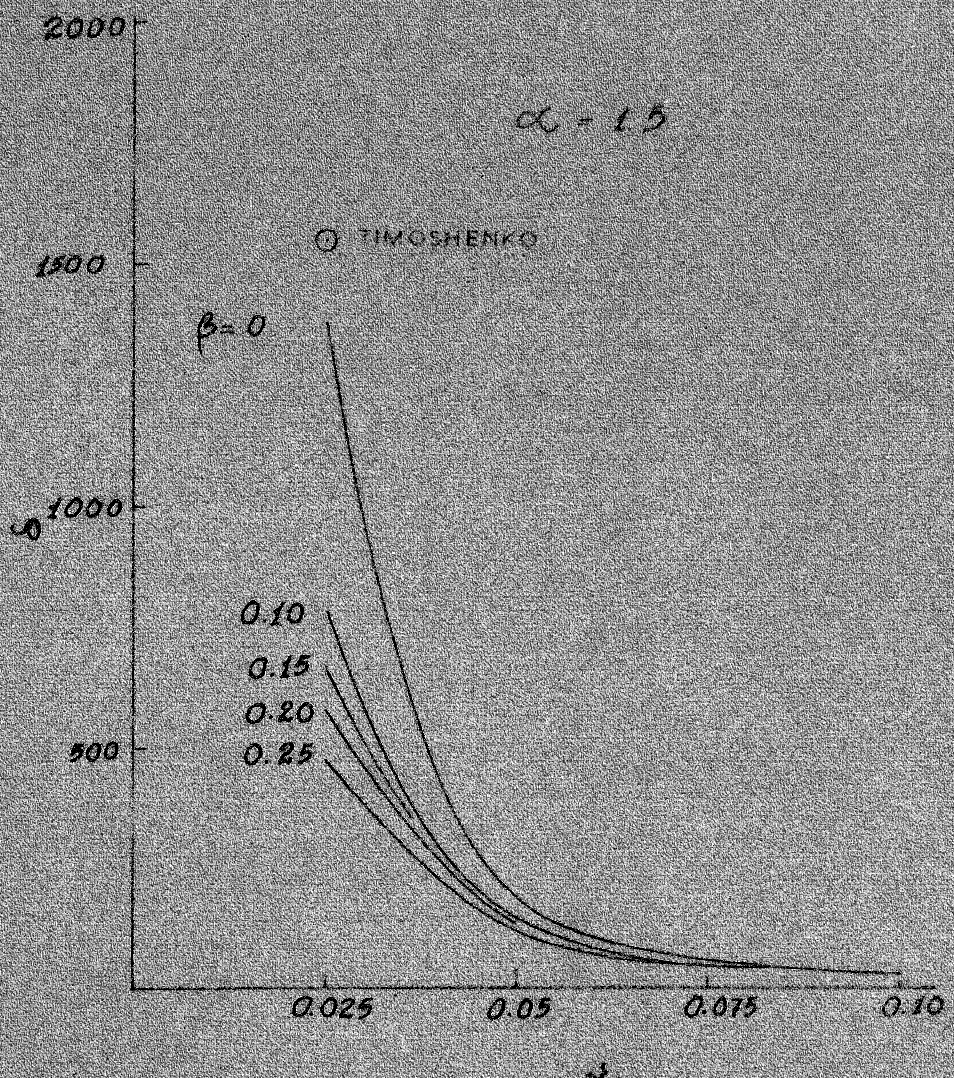


FIG. 3.5: DEFLECTION COEFFICIENTS FOR SIMPLY SUPPORTED SHELL.

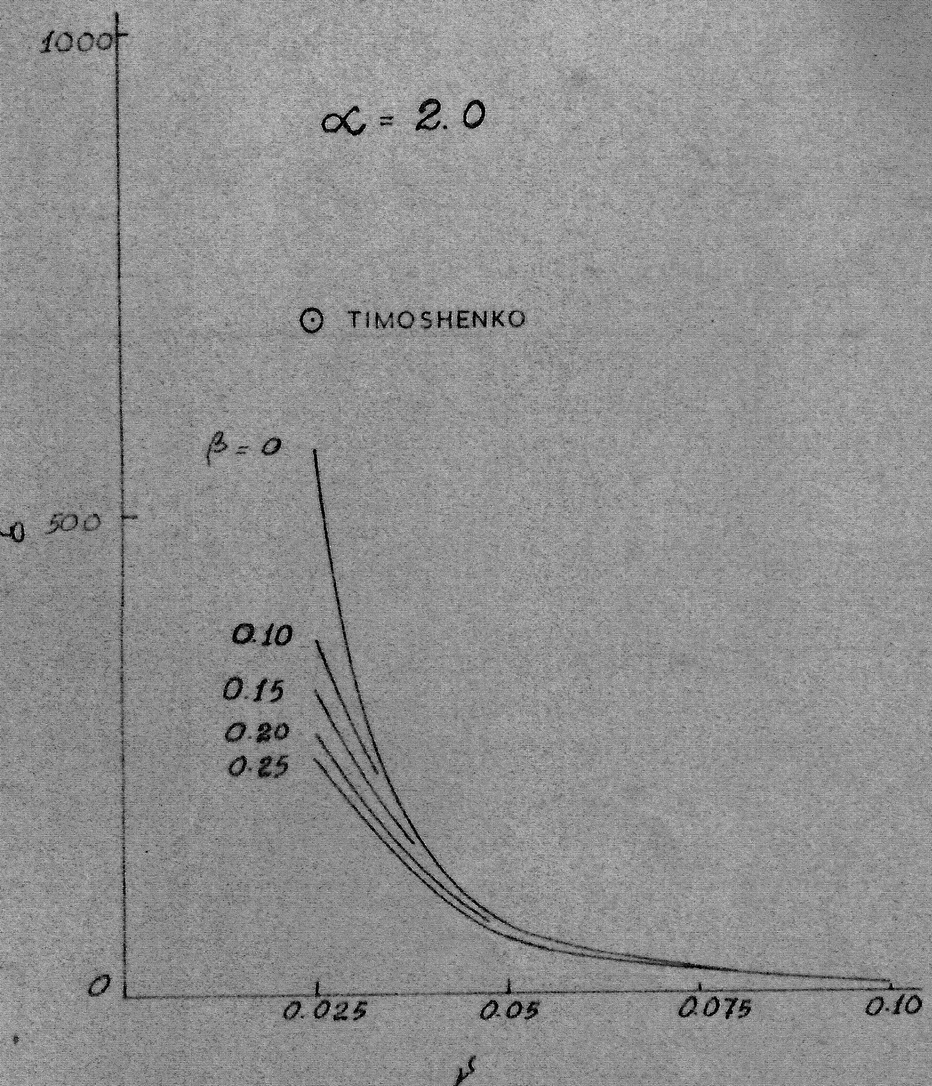


FIG. 3.6: DEFLECTION COEFFICIENTS FOR SIMPLY SUPPORTED SHELL.

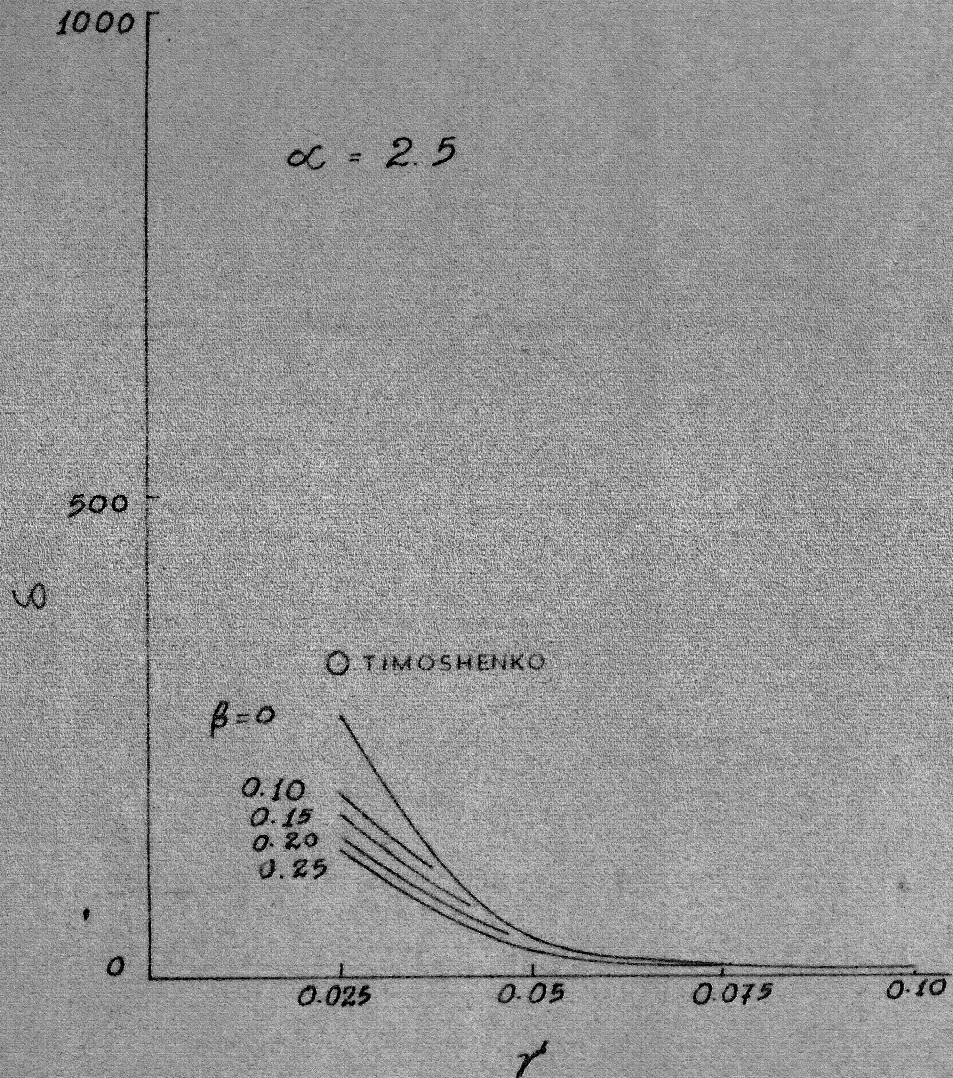


FIG. 3.7: DEFLECTION COEFFICIENTS FOR SIMPLY SUPPORTED SHELL.

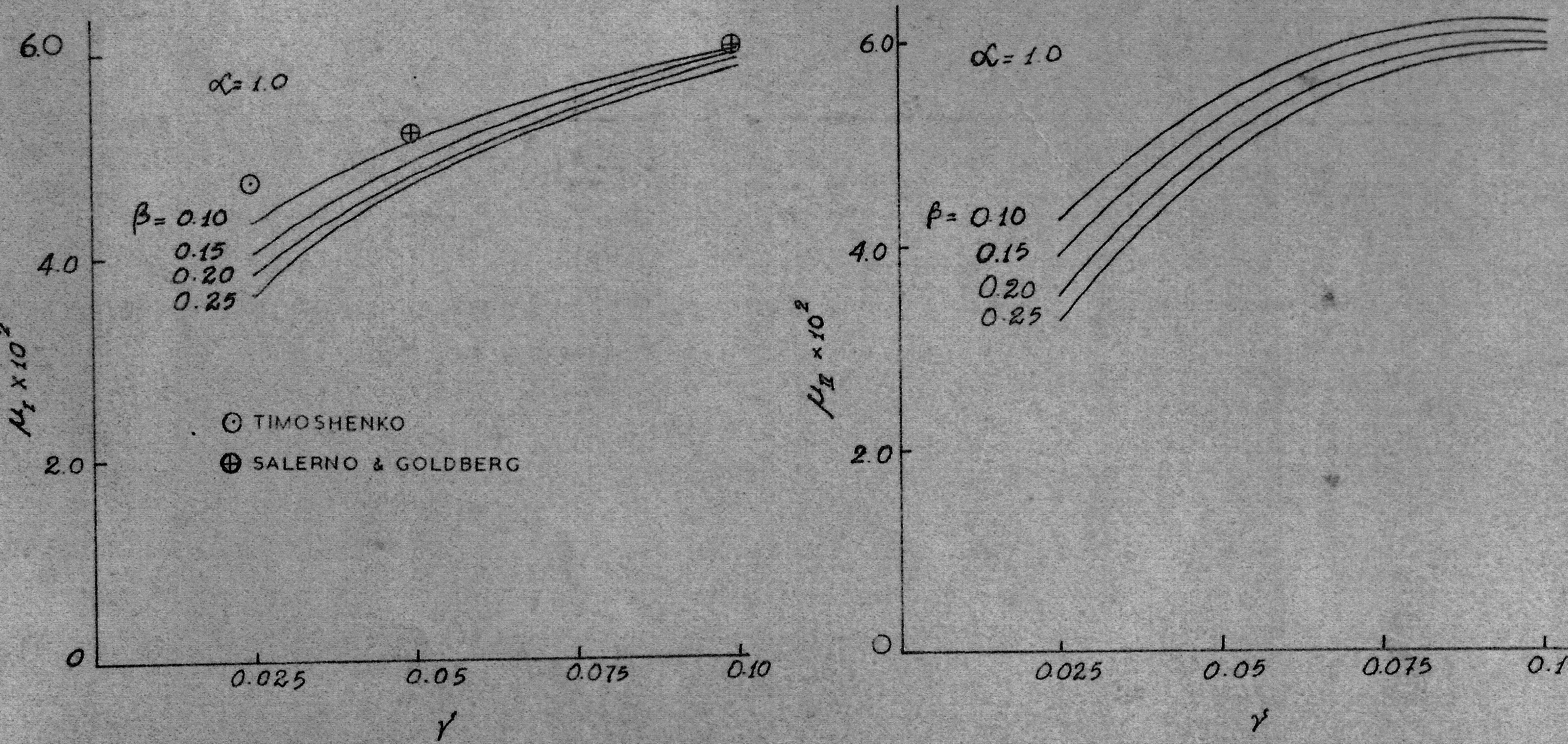


FIG. 3.8 & 3.9; MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED SHELL.

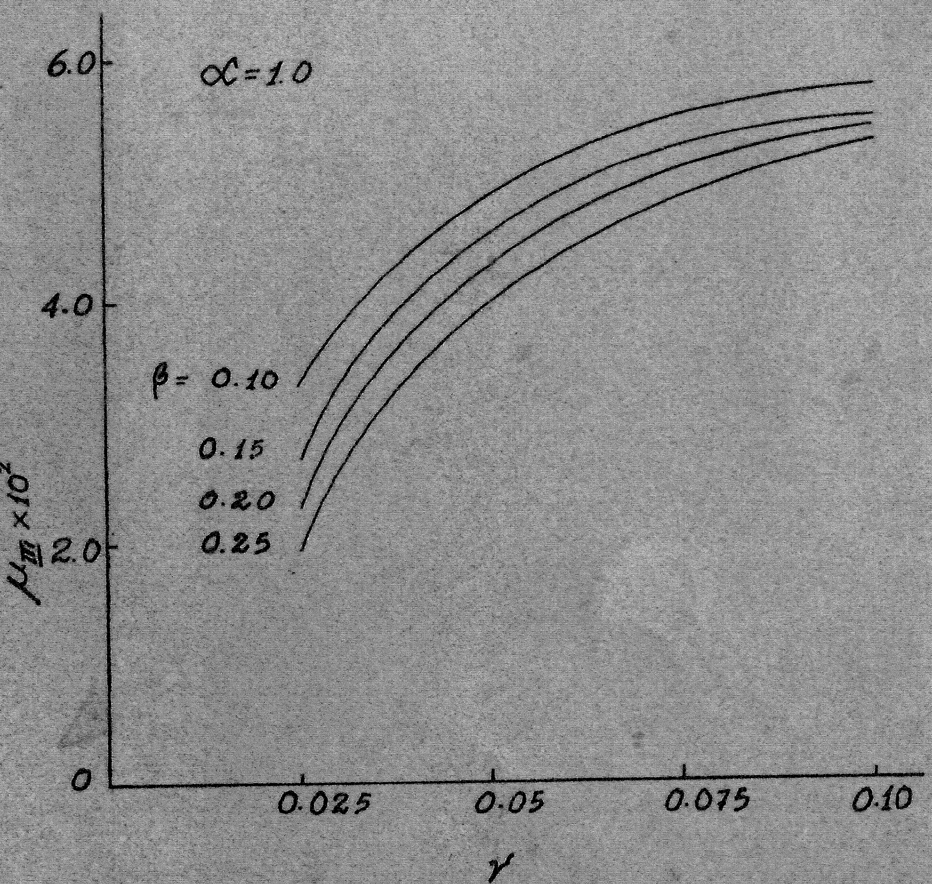
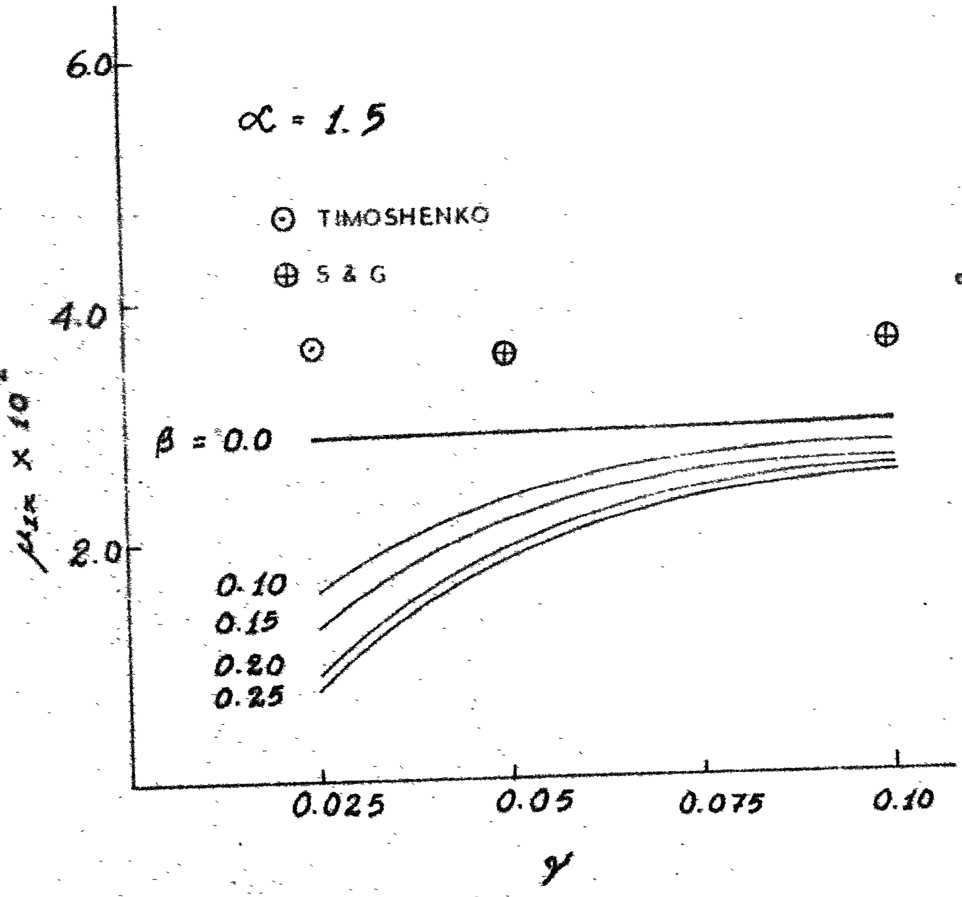


FIG. 3.10: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



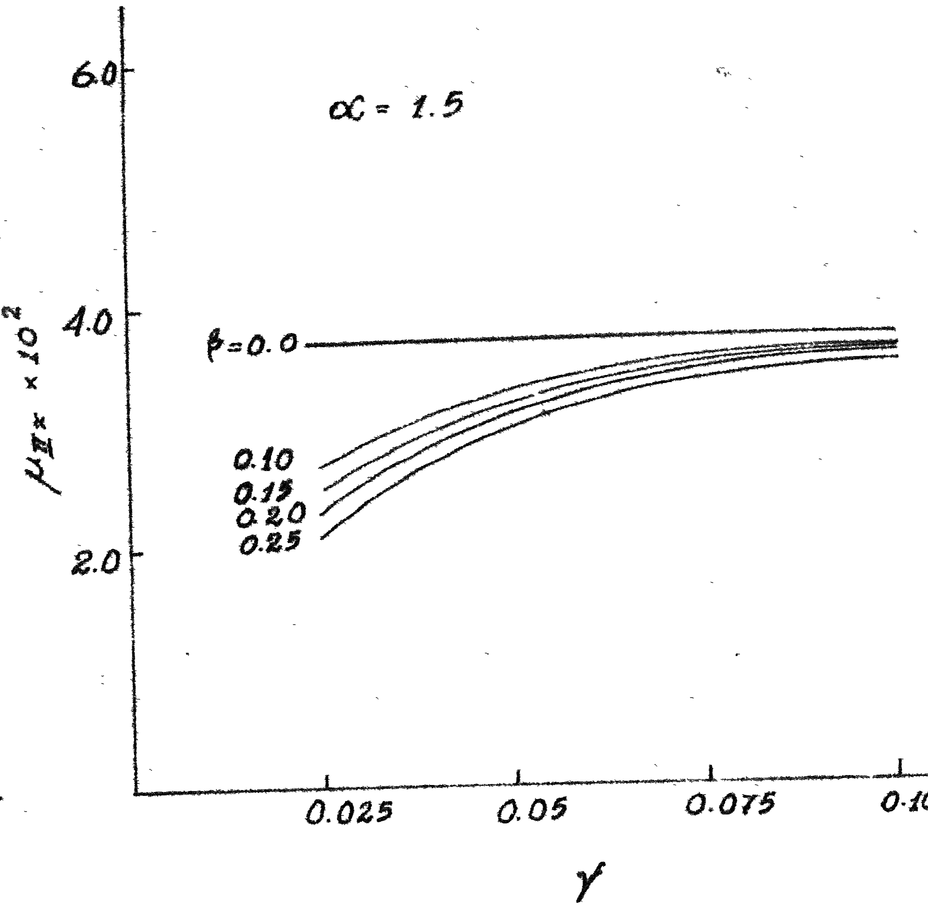
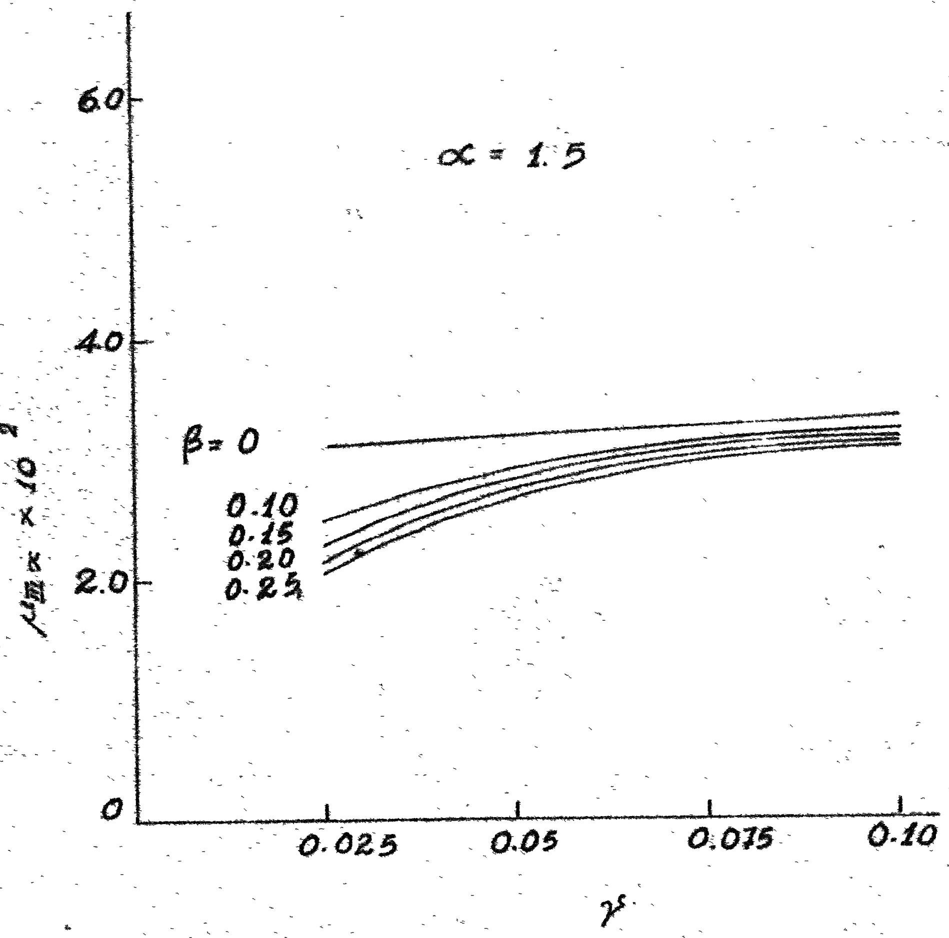


FIG. 3.11 & 3.12: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



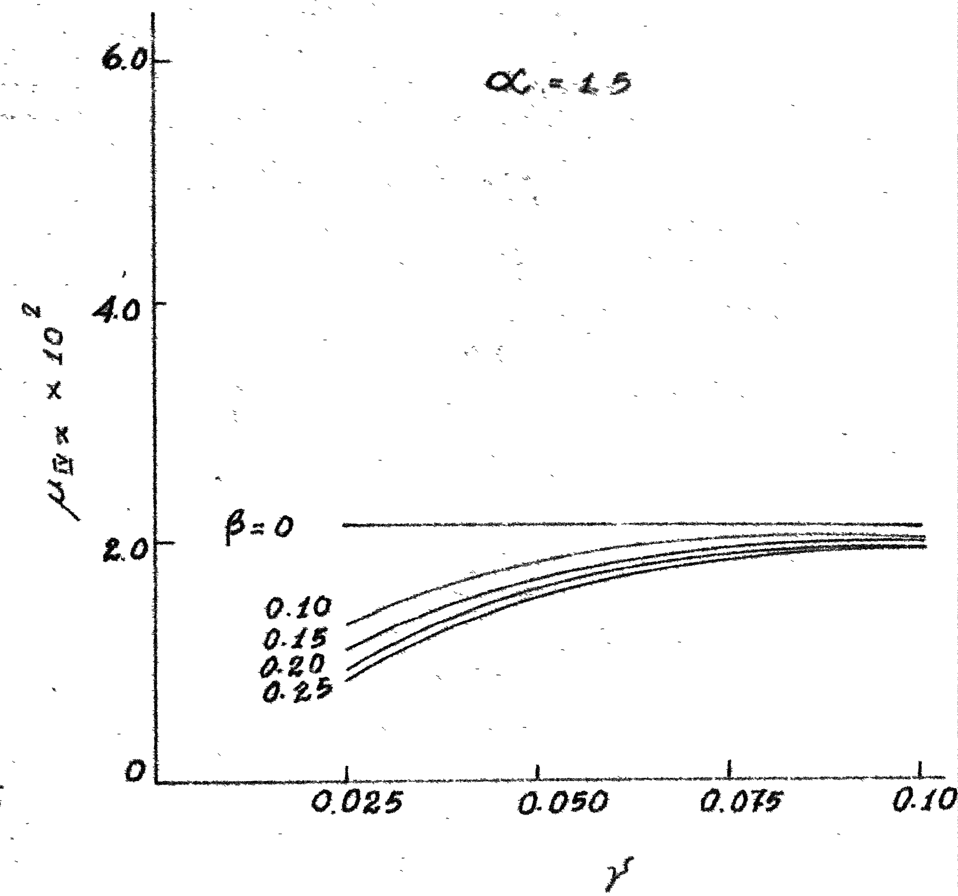
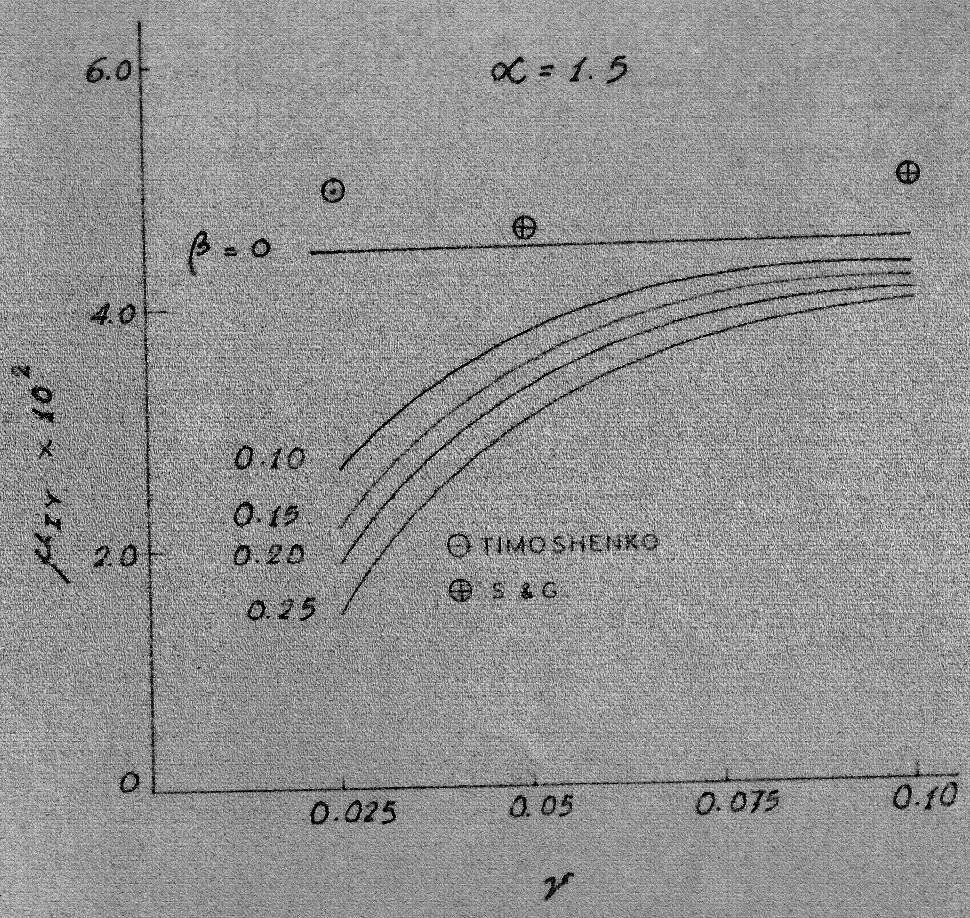


FIG. 3.13 & 3.14: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



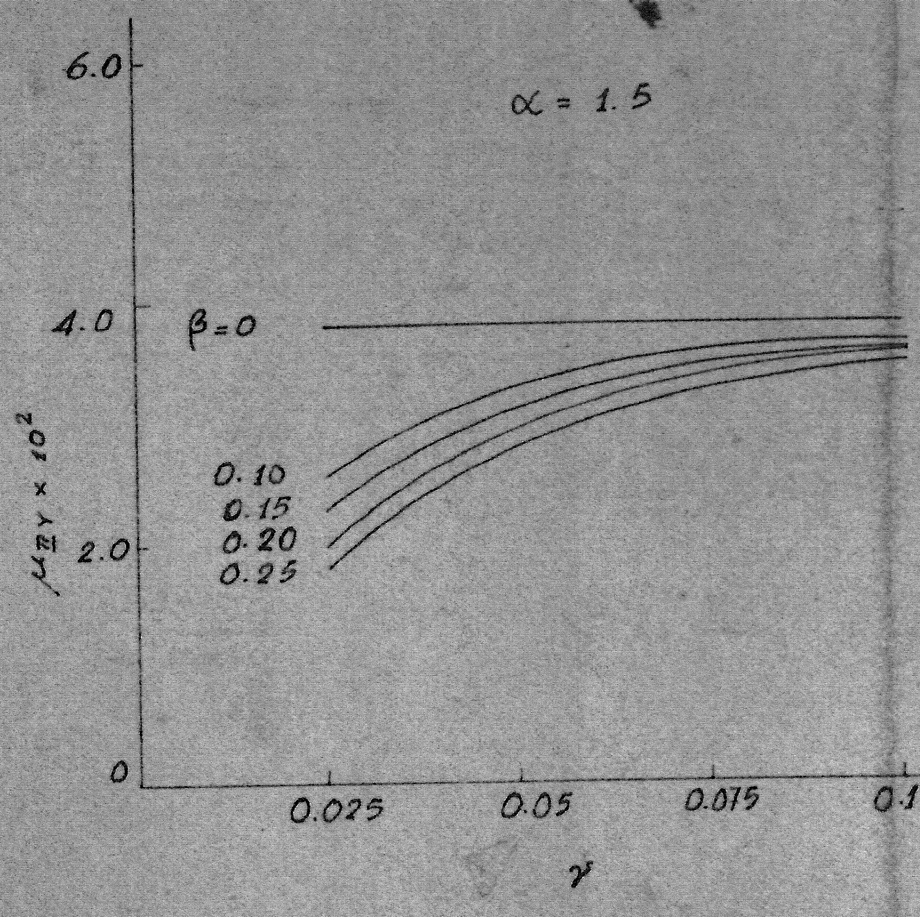
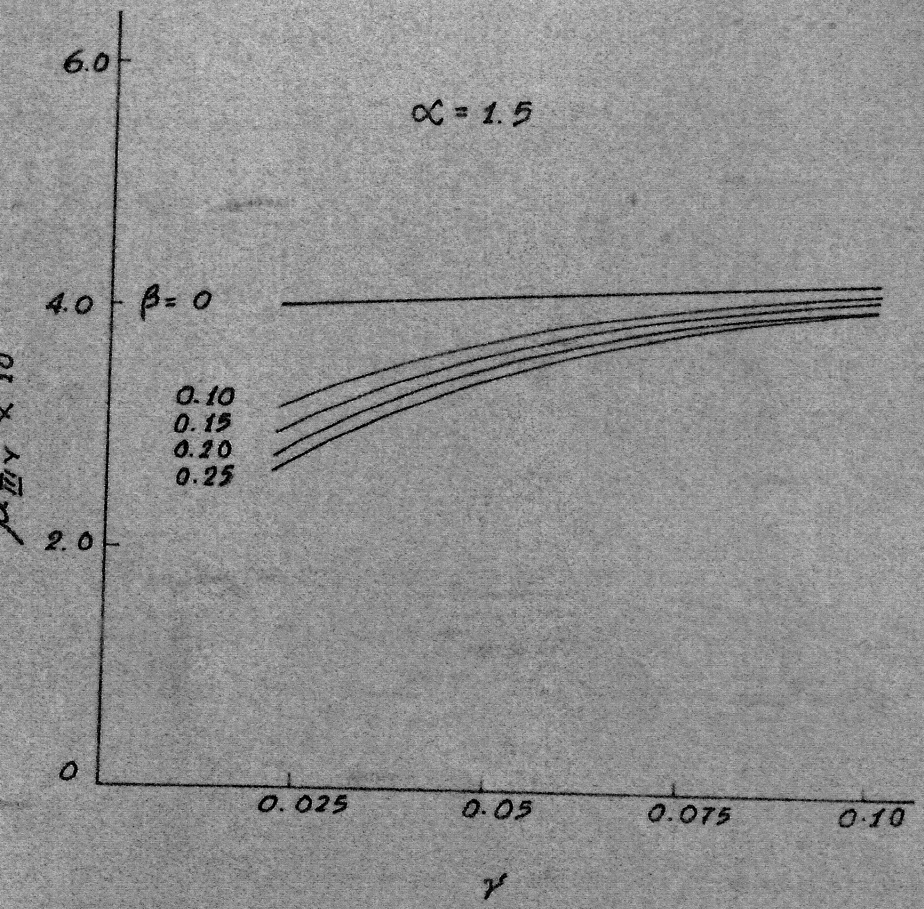


FIG. 3.15 & 3.16: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



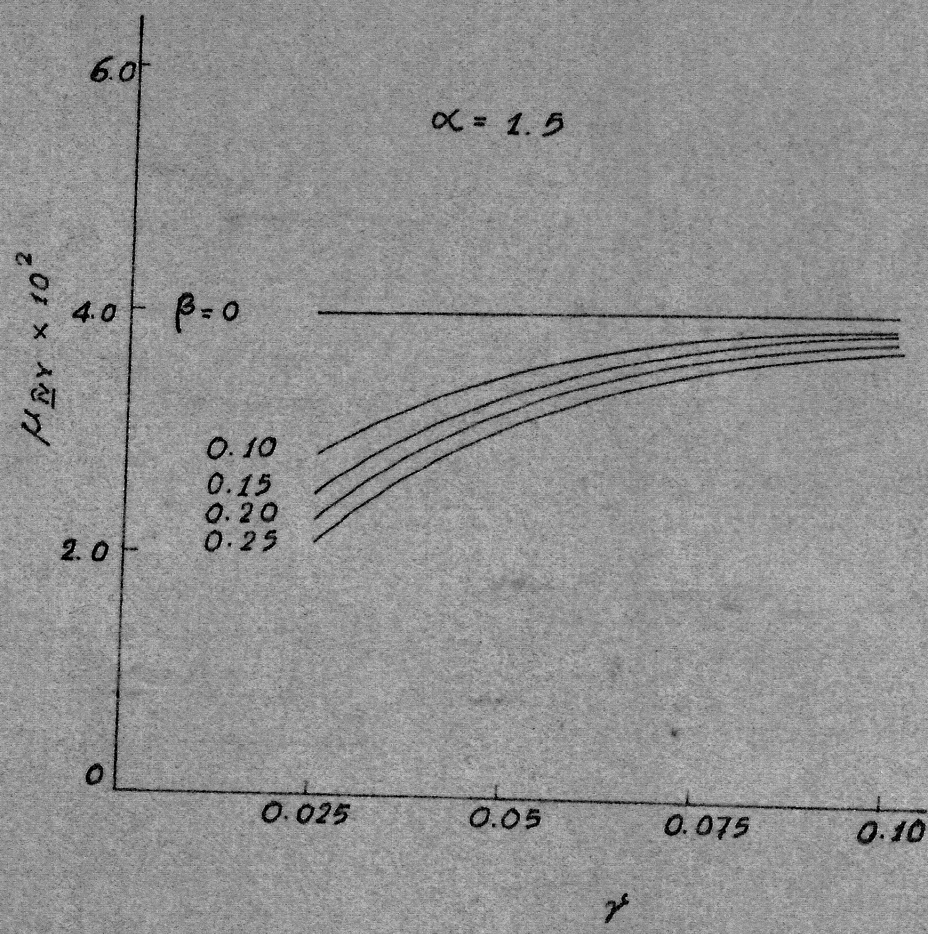


FIG. 3.17 & 3.18: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

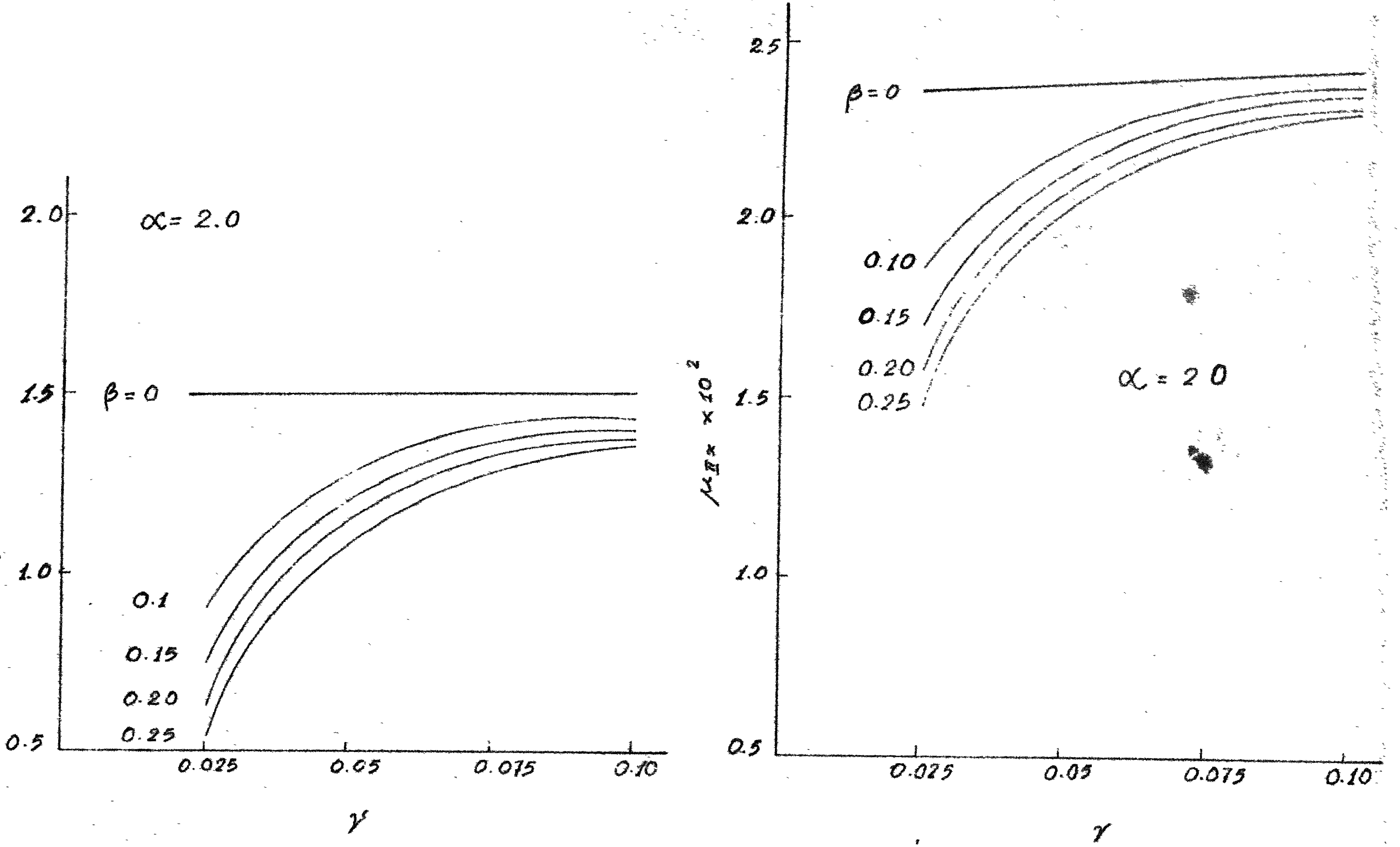


FIG. [illegible] & 3.25: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

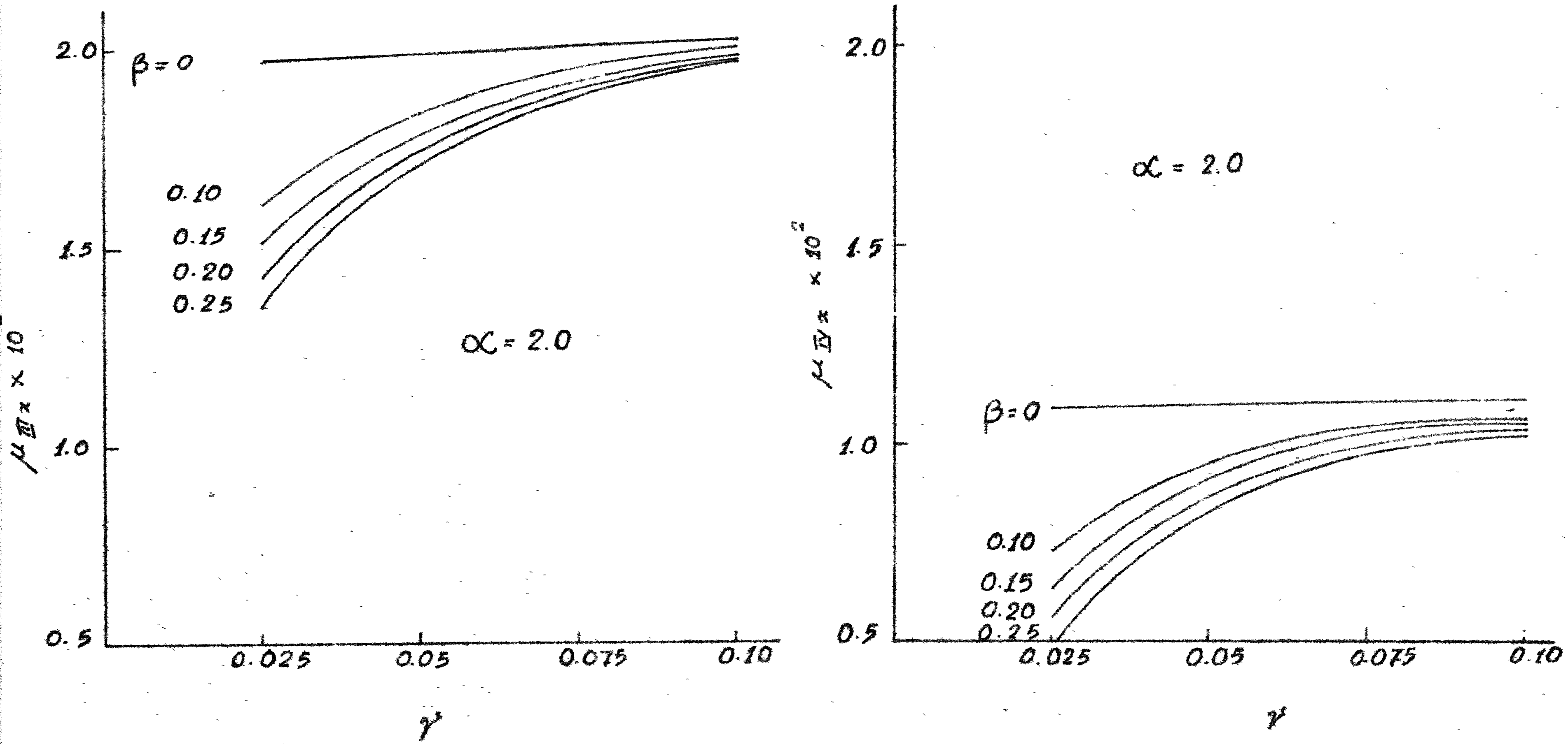


FIG. [illegible]: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

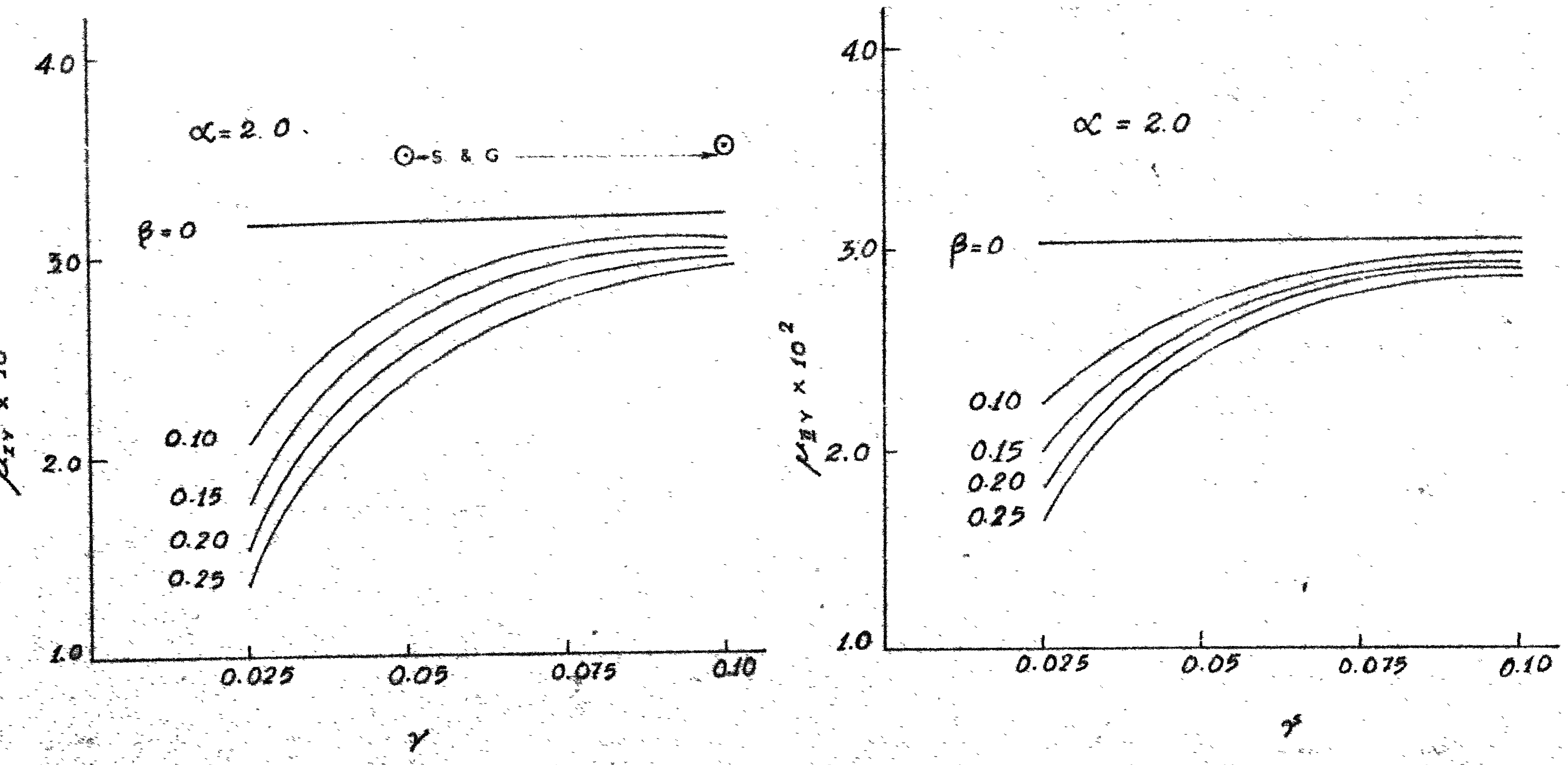
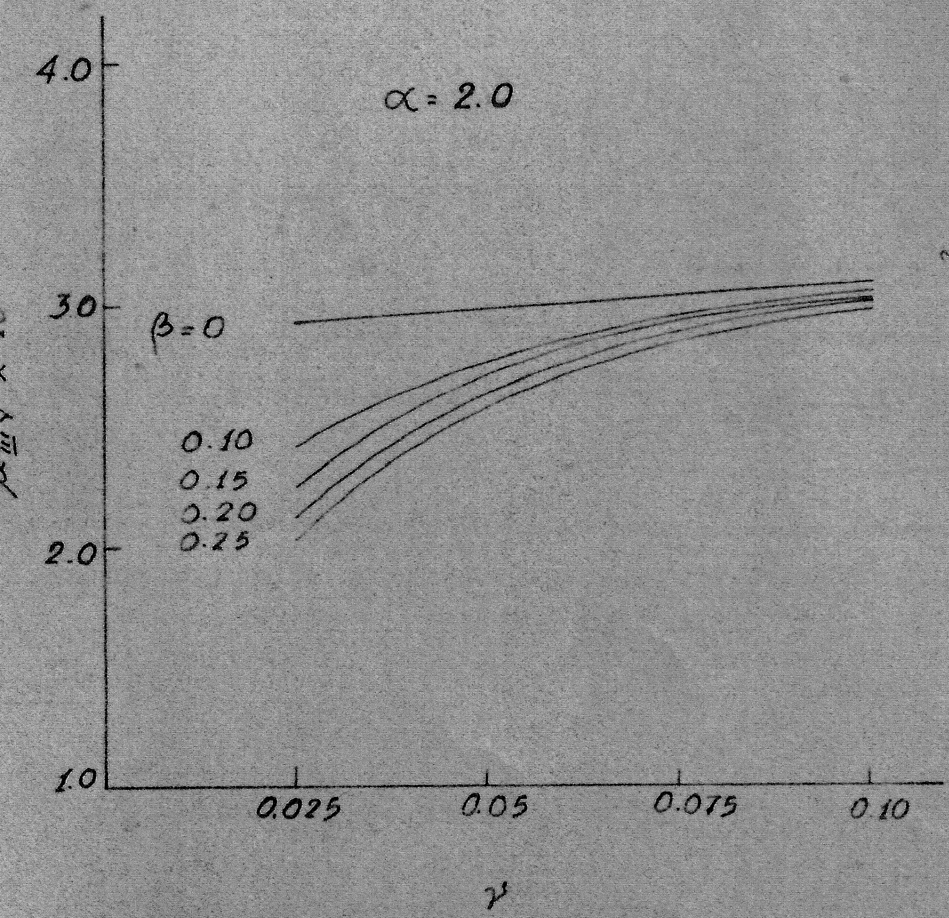


FIG. 3.23 & 3.24: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



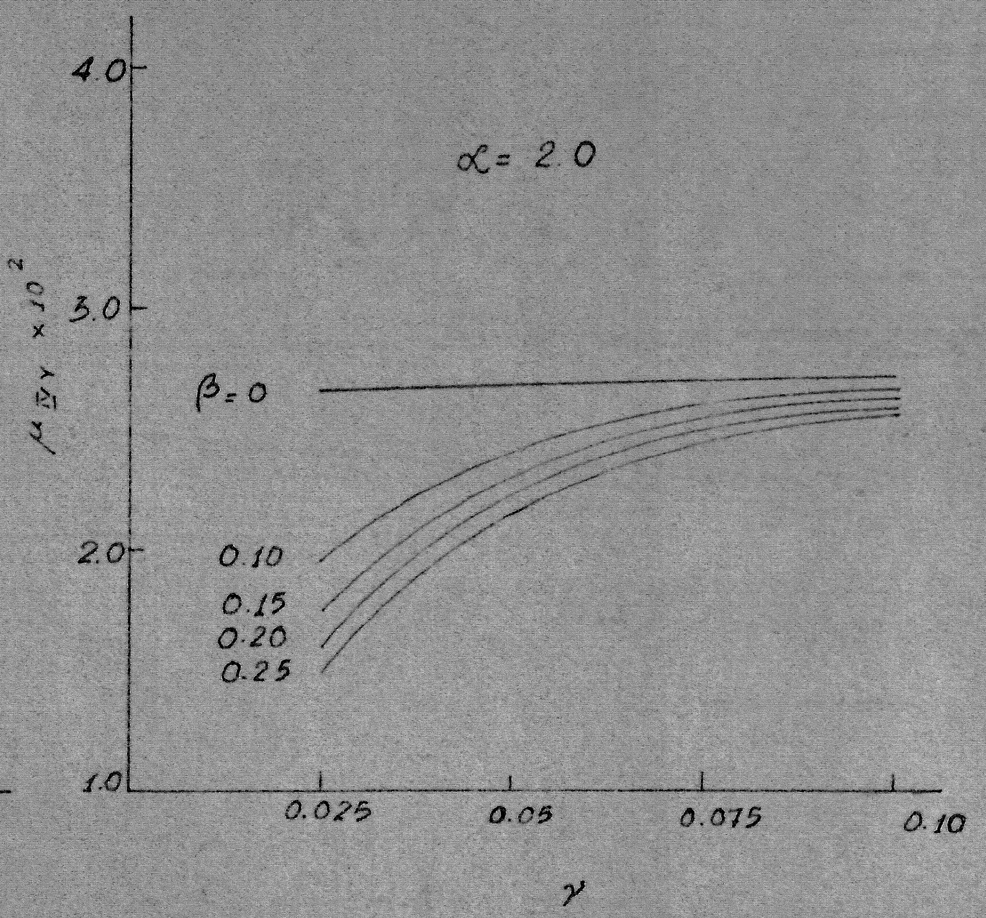


FIG. 3.25 & 3.26: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

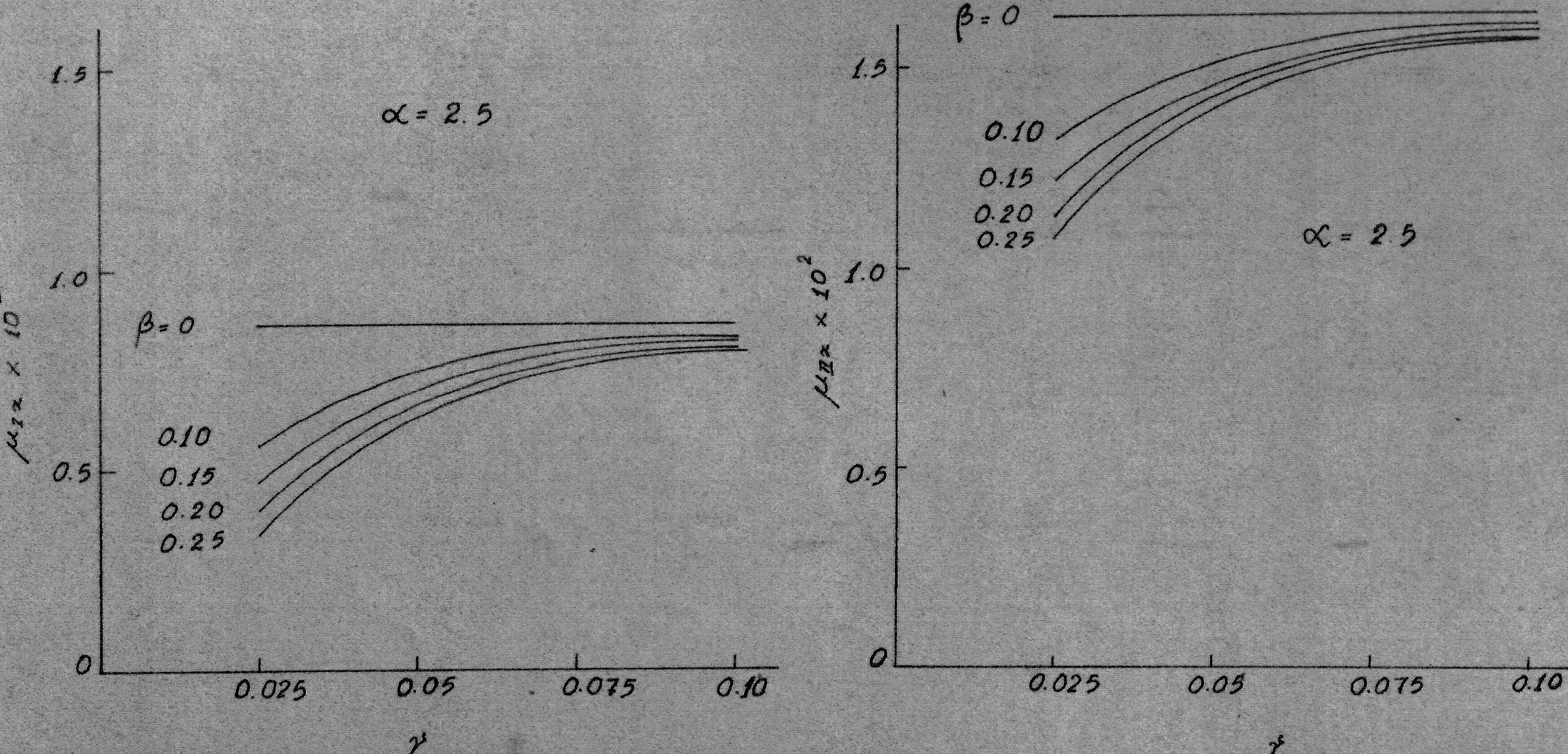
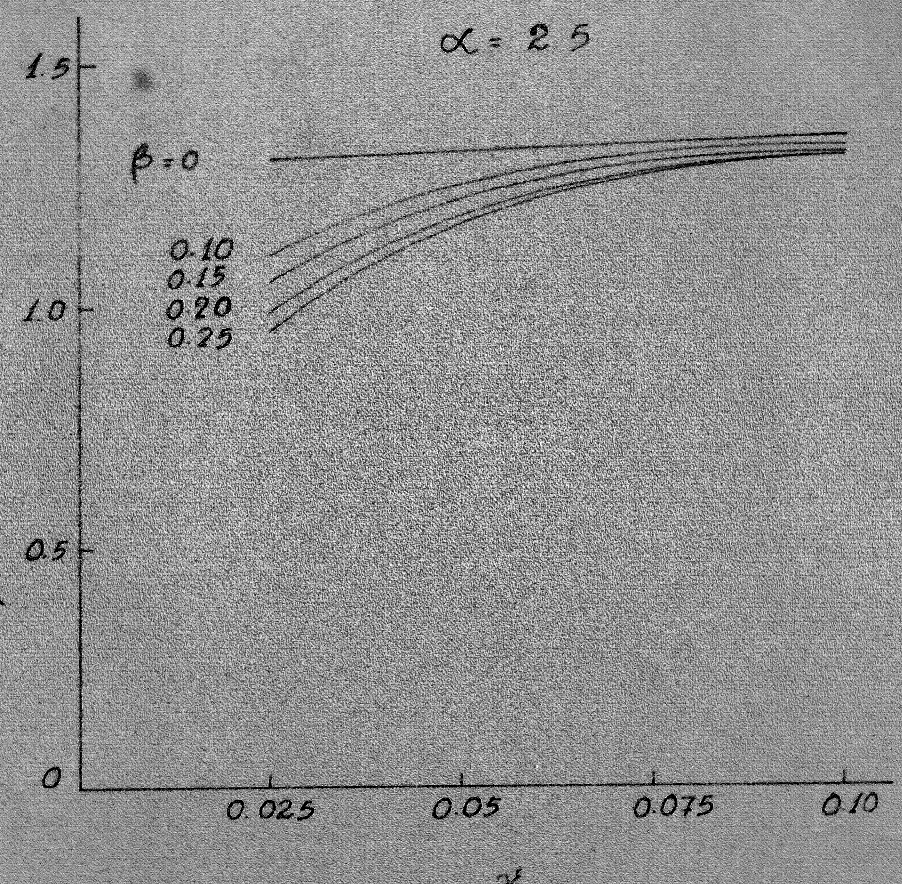


FIG. 3.27 & 3.28: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



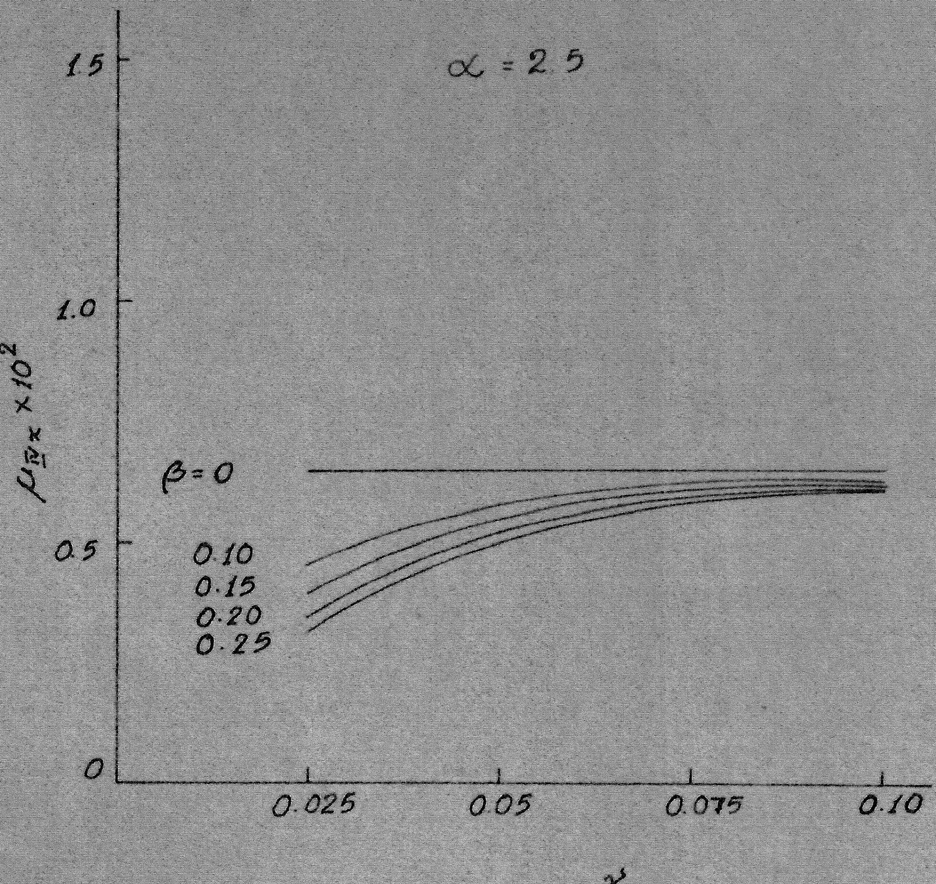


FIG. 3.29 & 3.30: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

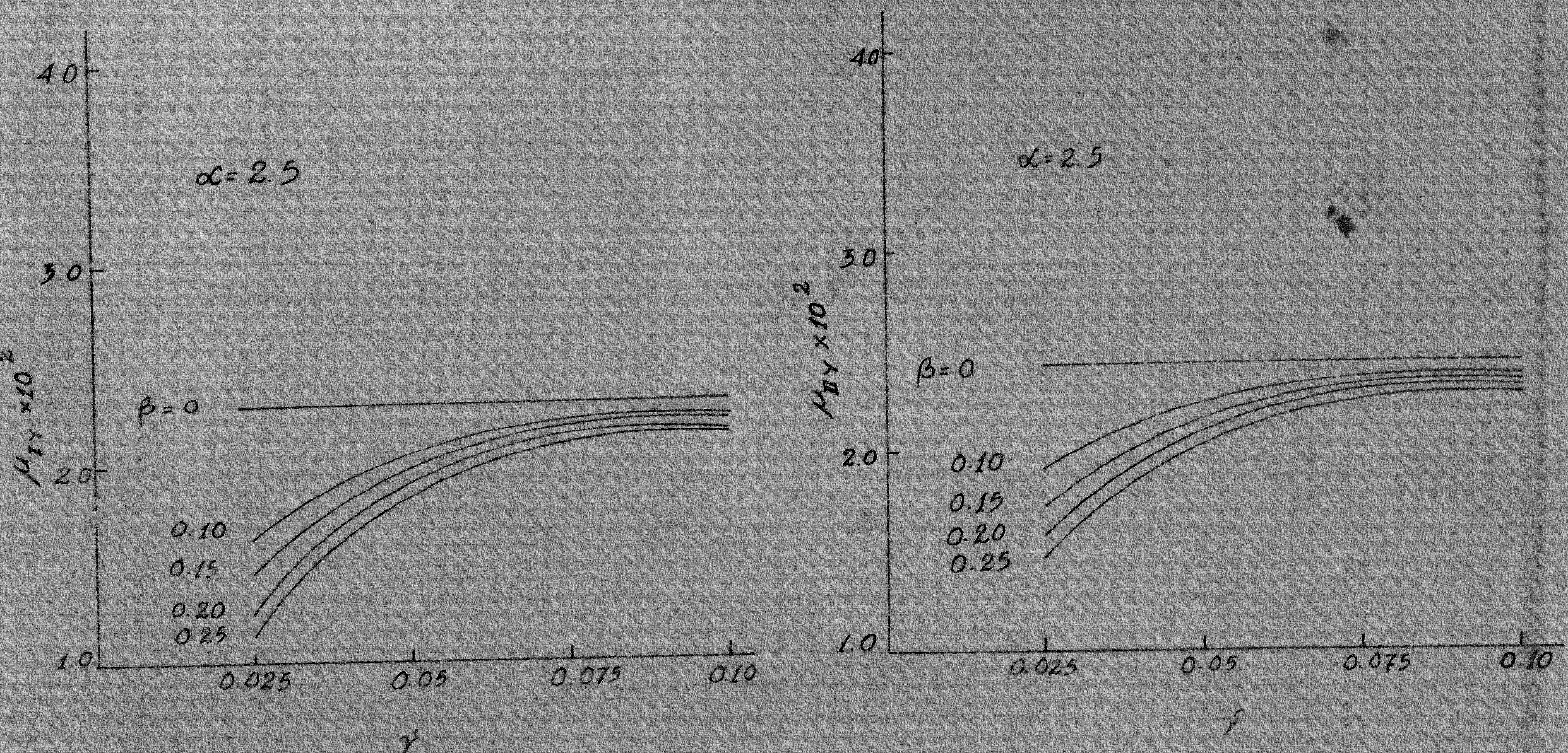
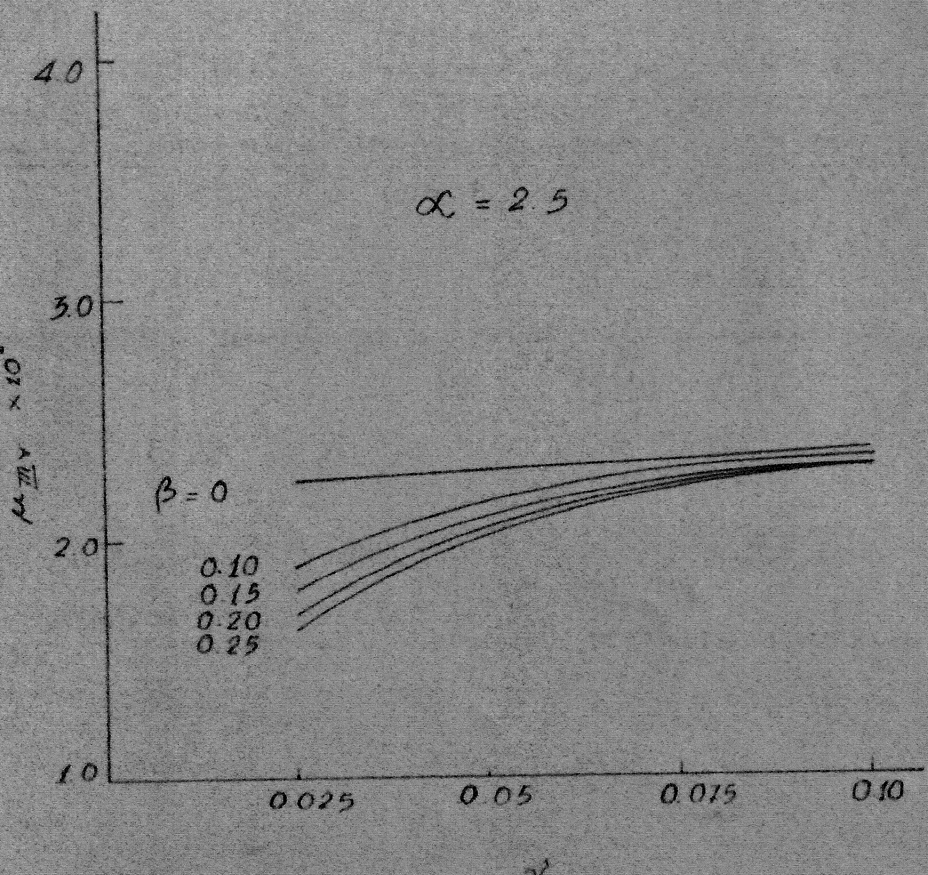


FIG. 3.31 & 3.32: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.



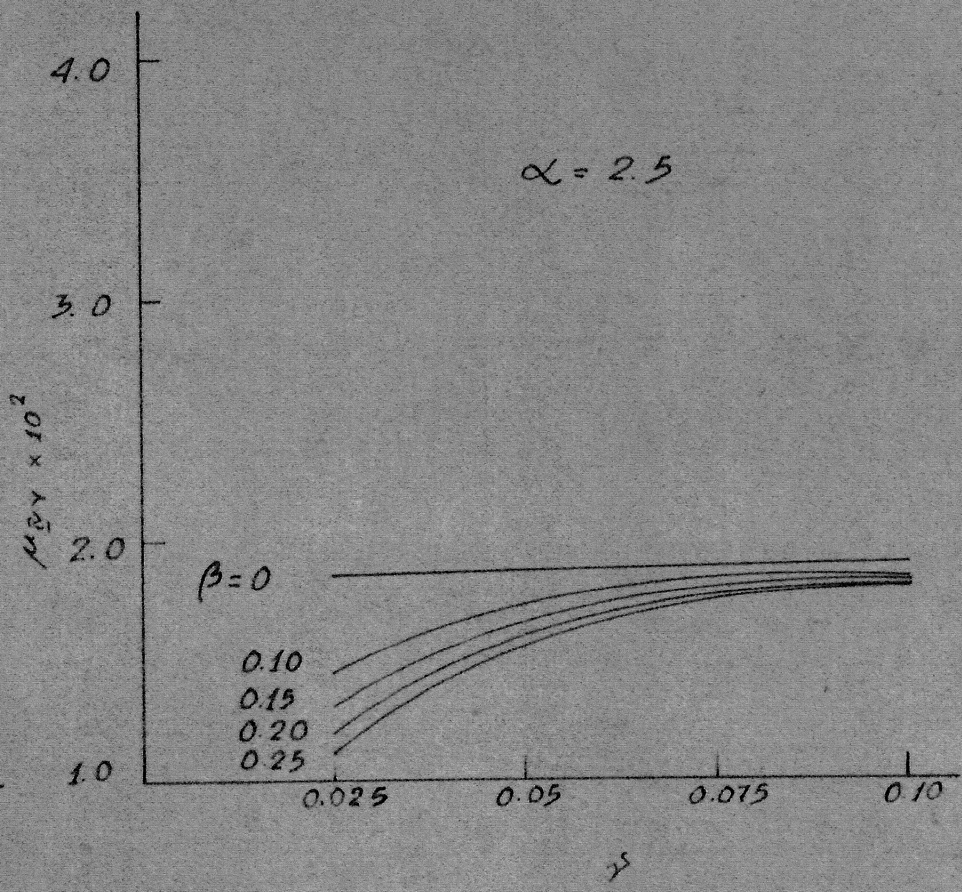


FIG. 3.33 & 3.34: MOMENT COEFFICIENTS FOR SIMPLY SUPPORTED HYPER SHELL.

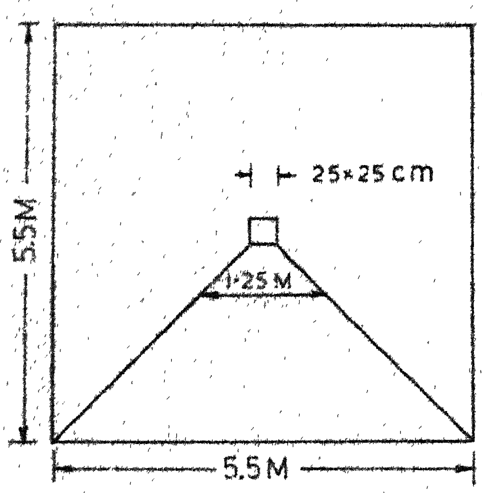


FIG. 335 FOUNDATION PLAN OF ILLUSTRATED EXAMPLE.

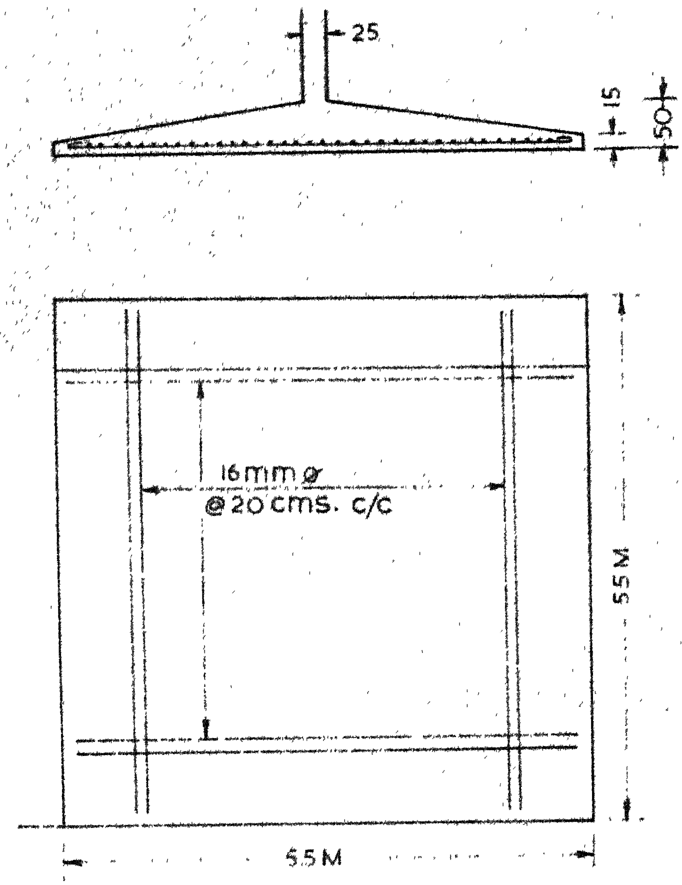


FIG. 3.36

FIG. 3.36 SPREAD FOOTING OF THE ILLUSTRATED EXAMPLE.

the central zone (zone I, Fig. 1.4).

One particular case of a shell with a fixed - fixed edge condition was numerically solved. The shell is of 4M x 6M in plan. The bending moment variation along the centre of the longer span is shown in Fig. 3.3. It is observed that the edge moment varies from 110 percent to 140 percent of the central moment for shell thickness ratio γ varying from 2.5 percent to 10 percent and for a shell rise of 25 percent.

3.6 CONCLUSION:

Since the assumption of simply supported boundary conditions can not exactly be realised in practice, this inherent limitation of the graphs restricts its use for design office practice. Nevertheless, this work points out several facts conclusively:

(i) The reduction in volume of concrete and steel is considerable, as the design example also illustrates.

(ii) Even for foundation the shell thickness is restricted more by practical considerations than by bending stresses, as is usual with thin shell design practice.

(iii) The area of steel is governed by the allowable requirements of the code of practice rather than the stresses occuring within.

(iv) Nothing has been said about secondary effects such as settlement, hydrostatic lift etc. which are vitally important in practice. So before its adoptation for design purpose a much more thorough and detailed investigation is suggested.

This work points out to the justification for adopting thin shell design method for foundation shells also, as is usually done. Only difference lies in the considerations of bending moments near the corners which may at times be high. This once again illuminates the path to be followed in future work.

TABLE 1

Deflection and Moment Coefficients for Simply supported Hyper shell.

$\alpha = 10$

β	γ	δ	μ_{I}	μ_{II}	μ_{III}	μ_{IV}
0.1	0.025	1640	4.37	4.33	3.30	4.33
	0.050	3.00	5.23	5.52	4.94	5.52
	0.075	108	5.63	5.96	5.56	5.96
	0.10	48	6.01	6.23	5.68	6.23
0.15	0.025	1310	4.06	3.88	2.69	3.88
	0.050	280	5.06	5.26	4.55	5.26
	0.075	100	5.63	5.80	5.24	5.80
	0.10	48	5.95	6.14	5.54	6.14
0.20	0.025	1150	3.83	3.55	2.27	3.55
	0.050	280	4.91	5.03	4.23	5.03
	0.075	102	5.43	5.67	5.03	5.67
	0.10	44	5.89	6.04	5.40	6.04
0.25	0.025	960	3.64	3.29	1.96	3.29
	0.050	260	4.77	4.83	3.94	4.83
	0.075	98	5.35	5.54	4.84	5.54
	0.10	43	5.82	5.96	5.27	5.96

Note: All μ -values should be multiplied by 10^{-2}.

TABLE 2

Deflection and Moment Coefficients for simply Supported Hypar shell.

$\alpha = 1.5$

β	γ	δ	μ_{Ix}	μ_{IIx}	μ_{IIIx}	μ_{IVx}	μ_{Iy}	μ_{IIy}	μ_{IIIy}	μ_{IVy}
0.0	0.025	1568	2.82	3.66	3.10	2.09	4.45	3.82	3.97	3.96
	0.050	172	2.82	3.67	3.13	2.10	4.46	3.83	4.02	3.99
	0.075	51	2.83	3.73	3.20	2.11	4.48	3.85	4.12	4.04
	0.10	22	2.83	3.78	3.30	2.12	4.49	3.86	4.26	4.12
0.10	0.025	191	1.61	2.74	2.50	1.32	2.66	2.59	3.17	2.80
	0.050	141	2.38	3.36	2.93	1.82	3.80	3.39	3.74	3.57
	0.075	46	2.61	3.56	3.10	1.97	4.15	3.63	3.97	3.83
	0.10	20	2.70	3.68	3.24	2.04	4.30	3.73	4.17	3.99
0.15	0.025	661	1.30	2.48	2.33	1.13	2.21	2.27	2.94	2.50
	0.050	131	2.20	3.23	2.84	1.71	3.55	3.22	3.62	3.40
	0.075	44	2.51	3.49	3.05	1.91	4.00	3.53	3.91	3.74
	0.10	19.7	2.64	3.64	3.21	2.00	4.21	3.67	4.13	3.93
0.20	0.023	570	1.09	2.28	2.19	0.98	1.90	2.03	2.77	2.28
	0.050	127	2.05	3.11	2.77	1.61	3.32	3.06	3.52	3.25
	0.075	43	2.42	3.42	3.00	1.85	3.87	3.44	3.85	3.65
	0.10	19.3	2.58	3.59	3.18	1.96	4.12	3.60	4.09	3.87
0.25	0.025	484	0.92	2.11	2.08	0.87	1.66	1.83	2.63	2.11
	0.05	116	1.92	3.01	2.70	1.53	3.12	2.92	3.43	3.13
	0.075	42	2.33	3.36	2.97	1.80	3.74	3.35	2.79	3.57
	0.10	19	2.52	3.55	3.15	1.92	4.03	3.55	4.05	3.81

Note: All μ values should be multiplied by 10^{-2}

TABLE 3

Deflection and Moment coefficients for simply supported hyper shell.

$\alpha = 2.0$

β	γ	δ	μ_{Ix}	μ_{IIx}	μ_{IIIx}	μ_{IVx}	μ_{Iy}	μ_{IIy}	μ_{IIIy}	μ_{IVy}
0.0	0.025	568	1.48	2.36	1.95	1.09	3.17	3.03	2.92	2.65
	0.050	71	1.48	2.37	1.95	1.10	3.18	3.04	2.95	2.66
	0.075	21	1.48	2.39	1.98	1.10	3.19	3.05	3.01	2.69
	0.10	9	1.49	2.42	2.03	1.11	3.20	3.06	3.09	2.73
0.10	0.025	368	0.90	1.86	1.61	0.73	2.08	2.23	2.41	1.95
	0.050	62	1.29	2.21	1.84	0.97	2.81	2.77	2.78	2.43
	0.075	20	1.39	2.31	1.93	1.04	3.00	2.92	2.93	2.58
	0.10	8.6	1.43	2.37	2.00	1.07	3.09	2.99	3.04	2.67
0.15	0.025	315	0.74	1.70	1.51	0.63	1.78	1.99	2.25	1.75
	0.050	60	1.20	2.14	1.80	0.92	2.65	2.66	2.71	2.33
	0.075	19	1.34	2.28	1.91	1.01	2.93	2.86	2.89	2.52
	0.10	8.5	1.40	2.35	1.98	1.05	3.04	2.95	3.02	2.63
0.20	9.025	277	0.62	1.57	1.42	0.55	1.55	1.81	2.12	1.60
	0.050	56	1.13	2.08	1.76	0.87	2.51	2.56	2.65	2.24
	0.075	19	1.30	2.24	1.88	0.99	2.85	2.81	2.85	2.48
	0.10	8.4	1.38	2.33	1.97	1.04	2.99	2.91	3.00	2.60
0.25	0.025	246	0.53	1.47	1.35	0.49	1.37	1.66	2.02	1.48
	0.050	53	1.06	2.02	1.72	0.83	2.39	2.47	2.59	2.16
	0.075	18	1.26	2.21	1.86	0.96	2.77	2.75	2.82	2.43
	0.10	8.3	1.35	2.30	1.96	1.02	2.93	2.88	2.98	2.57

Note: All μ -values should be multiplied by 10^{-2}.

TABLE 4

Deflection and Moment Coefficients for Simply Supported Hypar Shell.
$\alpha = 2.5$

β	ν	δ	μ_{Ix}	μ_{IIx}	μ_{IIIx}	μ_{IVx}	μ_{Iy}	μ_{IIy}	μ_{IIIy}	μ_{IVy}
0.0	0.025	269	0.86	1.62	1.30	0.64	2.28	2.42	2.22	1.84
	0.030	34	0.86	1.63	1.31	0.64	2.29	2.42	2.24	1.85
	0.075	10	0.86	1.64	1.33	0.64	2.29	2.43	2.28	1.87
	0.10	43	0.86	1.65	1.35	0.65	2.30	2.44	2.34	1.90
0.10	0.025	192	0.56	1.33	1.11	0.45	1.62	1.90	1.89	1.43
	0.050	31	0.76	1.54	1.25	0.58	2.08	2.26	2.14	1.72
	0.075	9.7	0.81	1.59	1.30	0.62	2.19	2.35	2.23	1.81
	0.10	4.2	0.83	1.63	1.34	0.63	2.24	2.39	2.31	1.86
0.15	0.025	169	0.47	1.23	1.03	0.40	1.42	1.72	1.78	1.30
	0.050	29.7	0.72	1.50	1.23	0.56	1.99	2.19	2.10	1.66
	0.075	9.4	0.79	1.57	1.29	0.60	2.15	2.32	2.21	1.78
	0.10	4.1	0.82	1.61	1.33	0.63	2.22	2.37	2.29	1.84
0.20	0.025	146	0.40	1.14	0.99	0.35	1.26	1.58	1.69	1.19
	0.050	27.8	0.69	1.46	1.20	0.53	1.90	2.12	2.05	1.61
	0.075	9.1	0.77	1.56	1.28	0.59	2.10	2.28	2.19	1.75
	0.10	4.4	0.81	1.60	1.32	0.62	2.19	2.35	2.28	1.82
0.25	0.025	131	0.34	1.07	0.95	0.31	1.13	1.46	1.61	1.11
	0.050	27.0	0.65	1.43	1.18	0.51	1.82	2.06	2.01	1.56
	0.075	9.1	0.75	1.54	1.26	0.58	2.06	2.25	2.16	1.72
	0.10	4.0	0.796	1.59	1.32	0.61	2.16	2.33	2.27	1.81

Note: All μ values should be multiplied by 10^{-2}.

NOTATION

x, y, z	=	Orthogonal cartesian coordinate system.
s, t	=	Surface coordinate system the x and y direction respectively.
$\bar{i}, \bar{j}, \bar{k}$	=	Unit vectors in the cartesian system.
$\bar{e}_s, \bar{e}_t, \bar{e}_\infty$	=	Unit vectors in s, t and normal to the surface respectively.
$\bar{r}$	=	Radius vector of a point from a chosen origin.
ω	=	Included angle between s and t direction.
R_s, R_t	=	Radius of curvature along s and t direction respectively.
h	=	Thickness of the shell.
T_{ij}	=	Membrane force in i-plane j - direction.
M_{ij}	=	Moment vector in i-plane j-direction following a right handed cork-screw rule.
$\bar{F}_i$	=	Force in i-plane
$\bar{M}_i$	=	Moment vector in i-plane
c	=	Rise of the shell.
a	=	Span in x direction (Longer)
b	=	Span in y direction (Smaller)
α	=	a/b = Span ratio
β	=	c/b = Rise ratio
γ	=	h/b = Thickness ratio
E	=	Modulus of elasticity of the material.
W	=	Complementary energy per unit volume.

σ_1, σ_2 = Normal stresses in 1 and 2 direction.

τ_{ij} = Shear stresses in i-plane, j - direction.

u, v, w = Displacements in x, y and z - direction.

ν = Poisson's ratio.

U = Strain energy per unit volume.

D = Flexural rigidity of a plate element.

$\bar{q}$ = External load vector

Any other notation used at any phase is defined before it is used.

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APPENDIX

In the main volume the strain-displacement relations have been assumed to write the energy equation in terms of displacements. This can easily be derived by writing down the unit vectors in s, t and ζ direction in the deformed state and then calculating the strains.

The surface is given by

$$z = z(x, y) \tag{A1}$$

Before deformation, a point in the shell can be denoted vectorially as:

$$r = x\bar{i} + y\bar{j} + z(x, y)\,\bar{k} \tag{A2}$$

Therefore, the unit vectors in s, t and ζ direction are obtained, in undeformed state, as (see equation 1.4.5).

$$\bar{e}_s = \frac{\bar{i} + z_x\,\bar{k}}{(1 + z_x{}^2)^{1/2}}$$

$$\bar{e}_t = \frac{\bar{j} + z_y\,\bar{k}}{(1 + z_y{}^2)^{1/2}} \tag{A3}$$

$$\bar{e}_\zeta = \bar{e}_s \times \bar{e}_t = \frac{\bar{k} - z_x\bar{i} - z_y\bar{j}}{(1+z_x^2)^{1/2}(1+z_y^2)^{1/2}}$$

and

$$\cos\omega = \bar{e}_s \cdot \bar{e}_t = \frac{z_x\, z_y}{(1+z_x^2)^{1/2}\,(1+z_y^2)^{1/2}}$$

$$\cos\theta = \bar{e}_s \cdot \bar{e}_s = 0 \tag{A4}$$

$$\cos\emptyset = \bar{e}_t \cdot \bar{e} = 0$$

where,

ω = angle between s and t direction before deformation

θ = $\pi/2$ = angle between s and ς direction before deformation

ϕ = $\pi/2$ = Angle between t and ς direction before deformation

After deformation the same quantities become

$$\bar{r}' = \bar{r} + \bar{U} = \left[x+u(x,y)\right] \bar{i} + \left[y+v(x,y)\right] \bar{j} + \left[z(x,y)+ w(x,y)\right] \bar{k}^{*} \tag{A5}$$

Therefore, the unit vectors, in the deformed state, are:

$$\bar{e}'_s = \frac{\partial \bar{r}'}{\partial x} \Big/ \left| \frac{\partial \bar{r}'}{\partial x} \right|$$

$$= \frac{(1+\frac{\partial u}{\partial x})\bar{i} + \frac{\partial v}{\partial x}\bar{j} + (z_x + \frac{\partial w}{\partial x})\bar{k}}{\left[(1+\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (z_x + \frac{\partial w}{\partial x})^2\right]^{1/2}}$$

$$\bar{e}'_t = \frac{\partial \bar{r}'}{\partial y} \Big/ -\left| \frac{\partial \bar{r}'}{\partial y} \right|$$

$$= \frac{\frac{\partial u}{\partial y}\bar{i} + (1+\frac{\partial v}{\partial y})\bar{j} + (z_y + \frac{\partial w}{\partial y})\bar{k}}{\left[(\frac{\partial u}{\partial y})^2 + (1+\frac{\partial v}{\partial y})^2 + (z_y + \frac{\partial w}{\partial y})^2\right]^{1/2}} \tag{A6}$$

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* The displacements are functions of x, y and z. But they can easily be reduced to functions of x and y for the case under consideration with the help of equation (A1).

$$\bar{e}'_{\zeta} = \frac{\partial \bar{r}'}{\partial z} \Big/ \left| \frac{\partial \bar{r}'}{\partial z} \right|$$

$$= \frac{z_y \left(1+\frac{\partial u}{\partial x}\right)+z_x \frac{\partial u}{\partial y}\, \bar{i} + z_y \frac{\partial v}{\partial x} + z_x \left(1+\frac{\partial v}{\partial y}\right) \bar{j} + z_y\left(z_x+\frac{\partial w}{\partial x}\right)+z_x\left(z_y+\frac{\partial w}{\partial y}\right) \bar{k}}{\left[z_y\left(1+\frac{\partial u}{\partial x}\right)+z_x \frac{\partial u}{\partial y}{}^2 + z_y \frac{\partial v}{\partial x} + z_x \left(1+\frac{\partial v}{\partial y}\right)^2 + \left\{z_y\left(z_x+\frac{\partial w}{\partial x}\right)+z_x\left(z_y+\frac{\partial w}{\partial y}\right)\right\}^2\right]^{1/2}}$$

The equation (A6) gets much more simplified after the non-linear terms are eliminated and usual shallow shell approximations are made. With these in mind (A6) reduces to

$$\bar{e}'_s = \frac{\left(1+\frac{\partial u}{\partial x}\right) \bar{i} + \frac{\partial v}{\partial x} \bar{j} + \left(z_x + \frac{\partial w}{\partial x}\right) \bar{k}}{\left(1+\frac{\partial u}{\partial x} + z_x \frac{\partial w}{\partial x}\right)}$$

$$\bar{e}'_t = \frac{\frac{\partial u}{\partial y} \bar{i} + \left(1+\frac{\partial v}{\partial y}\right) \bar{j} + \left(z_y + \frac{\partial w}{\partial y}\right) \bar{k}}{\left(1+\frac{\partial v}{\partial y} + z_y \frac{\partial w}{\partial y}\right)} \qquad \text{(A7)}$$

$$\bar{e}'_{\zeta} = \frac{z_y\left(1+\frac{\partial u}{\partial x}\right)+z_x \frac{\partial u}{\partial y}\ \bar{i} + z_y \frac{\partial v}{\partial x} + z_x\left(1+\frac{\partial v}{\partial y}\right) \bar{j} + z_y\left(\frac{\partial w}{\partial x} + z_x \frac{\partial w}{\partial y}\right. \bar{k}}{\left[z_y\left(1+2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + z_x\left(1+2\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y}\right)\right]^{1/2}}$$

and,

$$\text{Cos } \omega' = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z_x \frac{\partial w}{\partial y} + z_y \frac{\partial w}{\partial x}$$

$$\text{Cos } \theta' = z_x \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z_y \left(1+2\frac{\partial u}{\partial x}\right) \quad \text{(A10)}$$

$$\text{Cos } \phi' = z_x \left(1 + 2\frac{\partial v}{\partial y}\right) + z_y \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Now, for a shallow shell, the strain components are obtained as (equating ϵ_{zz} to zero):

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad , \quad \epsilon_{yy} = \frac{\partial v}{\partial y} \quad \text{(A11)}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + z_x \frac{\partial w}{\partial y} + z_y \frac{\partial w}{\partial x}$$

$$\epsilon_{xz} = z_x \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z_y \left(1+2\frac{\partial u}{\partial x}\right) \quad \text{(A12)}$$

$$\epsilon_{yz} = z_x \left(1+2\frac{\partial v}{\partial y}\right) + z_y \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

These are the strain-displacement relation used in this work to express energy in terms of displacements.